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Up-down asymmetric exact solutions to the Navier-Stokes equations

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ABSTRACT

Exact up-down asymmetric solutions to the Navier-Stokes equations for a viscous and incompressible fluid with time-dependent viscosity $\nu(t)$ are derived. Transformations of the exact solutions are defined that produce an infinite sequence of new solutions from each known one. The solutions are presented in terms of elementary functions and have no singularities. Three infinite-dimensional families of new exact axisymmetric unsteady solutions to the viscous magnetohydrodynamics equations are derived. Dynamics of vortex rings and vortex blobs is studied for some exact up-down asymmetric incompressible viscous fluid flows and viscous plasma flows.

I. INTRODUCTION

We derive new exact solutions to the Navier-Stokes equations[1] for a viscous and incompressible fluid

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nabla\Psi + \nu\nabla^2\mathbf{V}, \qquad \nabla\cdot\mathbf{V} = 0, \quad (1.1)$$

where $\mathbf{V}(\mathbf{x}, t)$ is the fluid velocity, $\Psi(\mathbf{x}, t)$ is the gravitational potential, $p(\mathbf{x}, t)$ is the pressure, ρ is the constant fluid density, and ∇^2 is the Laplace operator. As known,[2,3] the kinematic viscosity ν essentially depends on temperature. Since the latter depends on time t, we assume that viscosity $\nu = \nu(t)$ is an arbitrary piecewise continuous non-negative function of t.

Theory of Navier-Stokes equations for a viscous and incompressible fluid was developed in monographs[2–6] and in many other works. Several important exact solutions to the Navier-Stokes equations were found during the past century. The reviews[7–9] are devoted to both the steady and time-dependent exact solutions, see also Refs. 12 and 13.

In this paper, we introduce new axisymmetric unsteady exact solutions to the Navier-Stokes equations depending in the cylindrical coordinates r, z, and φ only on r and z. For the derived exact solutions, fluid velocity $\mathbf{V}(r, z, t)$ and its vorticity $\nabla \times \mathbf{V}(r, z, t)$ are not collinear and satisfy the equation $\nabla \times \mathbf{V} = \alpha\mathbf{V} - \alpha^2\xi r\hat{\mathbf{e}}_\varphi$, where α and ξ are arbitrary nonzero constants and $\hat{\mathbf{e}}_\varphi$ is the unit vector field in the φ-direction.

We construct an infinite-dimensional space of solutions for which fluid velocity $\mathbf{V}(r, z, t)$ is up-down asymmetric that means $\mathbf{V}(r, z, t)$ is not invariant under the reflection $z \to -z$. The solutions are analytic in the whole space $\mathbb{R}^3$ and are defined for all moments of time t. We present transformations acting on the space of exact solutions which generate an infinite sequence of new exact solutions from any known one.

We investigate dynamics in time t of the vortex rings and vortex blobs which are (for any fixed moment of time t_0) the maximal compact domains invariant with respect to the vorticity vector field $\nabla \times \mathbf{V}(r, z, t_0)$. We show that for the derived exact solutions to the Navier-Stokes equations, the vortex rings and vortex blobs are not frozen into the viscous fluid flow and collapse and disappear as $t \to \infty$.[28]

New exact axisymmetric unsteady solutions to equations of viscous and incompressible magnetohydrodynamics (MHD) are derived. The solutions describe three different regimes of plasma relaxation.

II. NEW EXACT Z-AXISYMMETRIC SOLUTIONS

A. An infinite family of exact solutions

Proposition 1. The z-axisymmetric incompressible fluid velocity fields

$$\mathbf{V}(r,z,t) = -\frac{1}{r}\frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha\psi_1}{r}\hat{\mathbf{e}}_\varphi \qquad (2.1)$$

represent exact solutions to the Navier-Stokes equations (1.1) *provided that the pressure* $p(r, z, t)$ *is defined by the formula*

$$p(r,z,t) = \rho\big[C + \Psi(r,z,t) + \alpha^2\xi\psi_1(r,z,t) - |\mathbf{V}(r,z,t)|^2/2\big], \tag{2.2}$$

and the stream function $\psi_1(r, z, t)$ *is an arbitrary solution to equations*

$$\frac{\partial^2\psi_1}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_1}{\partial r} + \frac{\partial^2\psi_1}{\partial z^2} = -\alpha^2\big[\psi_1 - \xi r^2\big], \tag{2.3}$$

$$\frac{\partial\psi_1}{\partial t} = -\alpha^2\nu(t)\big[\psi_1 - \xi r^2\big], \tag{2.4}$$

where α, ξ, *and* C *are arbitrary constant parameters and* $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, and $\hat{\mathbf{e}}_\varphi$ *are the unit vector fields tangent to the cylindrical coordinates* r, z, and φ.

Proof. Equation (2.1) implies $\nabla\cdot\mathbf{V}(r, z, t) = 0$. Therefore, vector fields $\mathbf{V}(r, z, t)$ (2.1) describe incompressible fluid flows. The corresponding vorticity field has the form

$$\nabla\times\mathbf{V}(r,z,t) = -\frac{\alpha}{r}\frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{\alpha}{r}\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_z - \frac{1}{r}\left(\frac{\partial^2\psi_1}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_1}{\partial r} + \frac{\partial^2\psi_1}{\partial z^2}\right)\hat{\mathbf{e}}_\varphi.$$

Substituting here Eq. (2.3), we get

$$\nabla\times\mathbf{V}(r,z,t) = \alpha\mathbf{V}(r,z,t) - \alpha^2\xi r\hat{\mathbf{e}}_\varphi. \tag{2.5}$$

Equation (2.5) yields

$$(\nabla\times\mathbf{V})\times\mathbf{V} = \left(\alpha\mathbf{V} - \alpha^2\xi r\hat{\mathbf{e}}_\varphi\right)\times\mathbf{V} = -\alpha^2\xi r\hat{\mathbf{e}}_\varphi\times\mathbf{V}. \tag{2.6}$$

Using here Eq. (2.1) and identities $\hat{\mathbf{e}}_\varphi\times\hat{\mathbf{e}}_r = -\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\varphi\times\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\varphi\times\hat{\mathbf{e}}_\varphi = 0$, we derive

$$\begin{aligned}(\nabla\times\mathbf{V})\times\mathbf{V} &= (-\alpha^2\xi r\hat{\mathbf{e}}_\varphi)\times\left[-\frac{1}{r}\frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha\psi_1}{r}\hat{\mathbf{e}}_\varphi\right]\\ &= -\alpha^2\xi\left[\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_z\right] = \nabla\big[-\alpha^2\xi\psi_1(r,z,t)\big].\end{aligned} \tag{2.7}$$

Applying the well-known identity $(\mathbf{V}\cdot\nabla)\mathbf{V} = (\nabla\times\mathbf{V})\times\mathbf{V} + \nabla\left(|\mathbf{V}|^2/2\right)$ and Eq. (2.7), we represent the Navier-Stokes equations (1.1) in the form

$$\frac{\partial\mathbf{V}}{\partial t} = -\nabla\left[\frac{1}{\rho}p - \Psi - \alpha^2\xi\psi_1(r,z,t) + \frac{1}{2}|\mathbf{V}|^2\right] + \nu(t)\nabla^2\mathbf{V}. \tag{2.8}$$

Substituting into (2.8) formula (2.2) for the pressure $p(r, z, t)$, we get

$$\frac{\partial\mathbf{V}}{\partial t} = \nu(t)\nabla^2\mathbf{V}. \tag{2.9}$$

Using identity $\nabla^2\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) - \nabla\times(\nabla\times\mathbf{V})$ and equations $\nabla\cdot\mathbf{V} = 0$, (2.5), and $\nabla\times(r\hat{\mathbf{e}}_\varphi) = 2\hat{\mathbf{e}}_z$, we find

$$\nabla^2\mathbf{V} = -\nabla\times(\alpha\mathbf{V} - \alpha^2\xi r\hat{\mathbf{e}}_\varphi) = -\alpha^2[\mathbf{V} - \xi(2\hat{\mathbf{e}}_z + \alpha r\hat{\mathbf{e}}_\varphi)]. \tag{2.10}$$

Using Eq. (2.4), we find the time-derivative of the vector field $\mathbf{V}$ (2.1) as

$$\begin{aligned}\frac{\partial\mathbf{V}}{\partial t} &= -\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial t}\right)\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial\psi_1}{\partial t}\right)\hat{\mathbf{e}}_z + \frac{\alpha}{r}\left(\frac{\partial\psi_1}{\partial t}\right)\hat{\mathbf{e}}_\varphi\\ &= -\alpha^2\nu(t)\left[-\frac{1}{r}\frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha\psi_1}{r}\hat{\mathbf{e}}_\varphi - \xi(2\hat{\mathbf{e}}_z + \alpha r\hat{\mathbf{e}}_\varphi)\right]\\ &= -\alpha^2\nu(t)[\mathbf{V} - \xi(2\hat{\mathbf{e}}_z + \alpha r\hat{\mathbf{e}}_\varphi)].\end{aligned} \tag{2.11}$$

Substitution of expressions (2.10) and (2.11) into Eq. (2.9) proves that it is identically satisfied. Therefore, formulas (2.1)–(2.4) define exact solutions to the Navier-Stokes equations (1.1). □

Remark 1. Equation (2.3) is a special nonhomogeneous case of the Grad-Shafranov equation[14,15]

$$\psi_{1rr} - \frac{1}{r}\psi_{1r} + \psi_{1zz} = -r^2\frac{dP}{d\psi_1} - G\frac{dG}{d\psi_1}, \tag{2.12}$$

corresponding to functions $P(\psi_1) = -\alpha^2\xi\psi_1$ and $G(\psi_1) = \alpha\psi_1$.

Equations (2.3) and (2.4) after substitution $\psi_1(r,z,t) = \xi r^2 + \tilde{\psi}(r,z,t)$ reduce to equations

$$\frac{\partial^2\tilde{\psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r} + \frac{\partial^2\tilde{\psi}}{\partial z^2} = -\alpha^2\tilde{\psi}, \quad \frac{\partial\tilde{\psi}}{\partial t} = -\alpha^2\nu(t)\tilde{\psi}. \tag{2.13}$$

The second one of Eq. (2.13) implies that the function $\tilde{\psi}(r,z,t)$ has the form $\tilde{\psi}(r,z,t) = f(t)\psi(r,z)$, where the time-dependent function $f(t)$ satisfies the equation

$$\frac{df(t)}{dt} = -\alpha^2\nu(t)f(t), \quad f(t) = \exp\left[-\alpha^2\int_0^t\nu(\tau)d\tau\right]. \tag{2.14}$$

Due to the first one of Eq. (2.13), the time-independent function $\psi(r, z)$ obeys the equation

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\alpha^2\psi. \tag{2.15}$$

The corresponding stream function $\psi_1(r, z, t)$ is

$$\psi_1(r,z,t) = \xi r^2 + f(t)\psi(r,z). \tag{2.16}$$

B. Transformations of exact solutions

Equation (2.15) is a special linear case of the Grad-Shafranov equation (2.12) for the axisymmetric plasma equilibria. Equation (2.15) is invariant under arbitrary shifts $z \to z + u_n$ and any differentiations $\partial^n/\partial z^n$. Therefore, for each concrete solution to Eq. (2.15), one can construct [using the linearity of Eq. (2.15)] an infinite sequence of exact solutions

$$\psi_{m,N}(r,z) = \sum_{n=1}^{m}a_n\psi(r,z+u_n) + \sum_{k=1}^{N}\sum_{n=1}^{m-1}b_{k,n}\frac{\partial^n\psi(r,z+z_{k,n})}{\partial z^n},$$

where a_n, u_n, and $b_{k,n}$ are arbitrary constants. The corresponding exact solutions to Eqs. (2.3) and (2.4) have the form

$$\psi_{1m,N}(r,z,t) = \xi r^2 + f(t)\left[\sum_{n=1}^{m}a_n\psi(r,z+u_n) + \sum_{k=1}^{N}\sum_{n=1}^{m-1}b_{k,n}\frac{\partial^n\psi(r,z+z_{k,n})}{\partial z^n}\right]. \tag{2.17}$$

Equation (2.15) has the following well-known exact solution

$$\psi(r,z) = -r^2G_2(\alpha R) = -\frac{r^2}{\alpha^2R^2}\left[\cos(\alpha R) - \frac{\sin(\alpha R)}{\alpha R}\right], \qquad (2.18)$$

where $R \equiv \sqrt{r^2+z^2}$ and $G_2(u) \equiv u^{-2}(\cos u - u^{-1}\sin u)$. The exact solution (2.18) was first discovered in hydrodynamics by Hicks[16,17] in 1899[29] and 57 years later was rediscovered in plasma physics by Chandrasekhar[19] and Woltjer[20] as a model of axisymmetric plasma equilibria. The corresponding fluid (or plasma) flow is now called the spheromak field. Moduli spaces of vortex knots for the spheromak vector field in different invariant domains were presented in Ref. 21, and for other solutions to Eq. (2.15) in Ref. 22. The safety factor for axisymmetric flows of barotropic gas and ideal incompressible fluid was studied in Ref. 23.

The vector field $\mathbf{V}_1(r, z, t)$ (2.1) with the stream function

$$\psi_1(r,z,t) = \xi r^2 - f(t)r^2G_2(\alpha R) \qquad (2.19)$$

has the form

$$\begin{aligned}\mathbf{V}_1(r,z,t) &= \alpha^2 rzf(t)G_3(\alpha R)\hat{\mathbf{e}}_r\\ &+\left[2\xi - f(t)(2G_2(\alpha R)+\alpha^2r^2G_3(\alpha R))\right]\hat{\mathbf{e}}_z\\ &+\alpha r[\xi - f(t)G_2(\alpha R)]\hat{\mathbf{e}}_\varphi, \end{aligned}\qquad (2.20)$$

where the function $G_3(u)$ is $G_3(u) \equiv u^{-4}[(3-u^2)u^{-1}\sin u - 3\cos u]$ and $f(t)$ is the function (2.14). The fluid velocity (2.20) together with the pressure

$$\begin{aligned}p(r,z,t) = \rho\Big[&C + \Psi(r,z,t)\\ &+\alpha^2r^2\xi[\xi - f(t)G_2(\alpha R)] - \frac{1}{2}|\mathbf{V}_1(r,z,t)|^2\Big]\end{aligned}$$

forms the new exact solution to the Navier-Stokes equations (1.1).

The poloidal sections of the stream surfaces $\psi_1(r, z, t)$ = const for any fixed time t coincide with trajectories of the fluid in variables r and z defined by the vector field $\mathbf{V}_1(r, z, t)$ (2.20). The stream function (2.19) is evidently invariant under the reflection $z \to -z$. Therefore, the poloidal sections $\psi_1(r, z, t)$ = const also are invariant. In this sense, the exact solutions (2.19) and (2.20) are up-down symmetric.

Remark 2. If $f(t) \to f_0$ = const as $t \to \infty$, then the viscous flow (2.20) tends to the steady solutions to Euler equations for the ideal incompressible fluid that were studied first by Hicks in Ref. 16 and later in Refs. 24 and 25. If $f(t) \to 0$ as $t \to \infty$ (for example, if $\nu \geq \nu_0$ = const > 0 or $\nu(t) \geq \nu_0/t$), then the derived exact solutions with stream functions $\psi_{1m,N}(r, z, t)$ (2.17) tend to the steady flow $\mathbf{V}(r,z) = \alpha\xi r\hat{\mathbf{e}}_\varphi + 2\xi\hat{\mathbf{e}}_z$ that has a constant vorticity $\nabla\times\mathbf{V}(r,z) = 2\alpha\xi\hat{\mathbf{e}}_z$ and hence has no vortex rings and blobs (see their definition in Sec. I). Therefore, the vortex rings and blobs for the exact flow (2.20) collapse and disappear as $t \to \infty$.

III. DYNAMICS OF VORTEX RINGS AND BLOBS FOR SOME UP-DOWN ASYMMETRIC EXACT FLUID FLOWS

A. Vortex rings and vortex blobs

In this section, we study some exact up-down asymmetric viscous flows. The stream functions $\psi_{1m,N}(r, z, t)$ (2.17) are up-down asymmetric if at least one coefficient $b_{k,n} \neq 0$ for $n = 2\ell + 1$. Therefore, the generic stream functions (2.17) and the corresponding exact fluid flows (2.1) are up-down asymmetric.

Equation (2.1) implies that, for any fixed moment of time t_0, the surface $\psi_1(r, z, t_0)$ = const (the angle $\varphi \in \mathbb{S}^1$ is arbitrary, $0 \leq \varphi < 2\pi$; here and below $\mathbb{S}^1$ is the unit circle) is an invariant submanifold for the vorticity vector field $\nabla\times\mathbf{V}(r, z, t_0)$. This follows from formula (2.5): $\nabla\times\mathbf{V} = \alpha\mathbf{V} - \alpha^2\xi r\hat{\mathbf{e}}_\varphi$, and the z-axisymmetry of the flow.

Since the surface $\psi_1(r, z, t_0)$ = const is z-axisymmetric, it is a disjoint union of either some spheres $\mathbb{S}^2$ or some tori $\mathbb{T}^2 = C_{\psi_1(t_0)} \times \mathbb{S}^1$ or some cylinders $\mathbb{C}^2 = R_{\psi_1(t_0)} \times \mathbb{S}^1$. Here, $C_{\psi_1(t_0)}$ and $R_{\psi_1(t_0)}$ are the level curves $\psi_1(r, z, t)$ = const in the poloidal plane (r, z) for a fixed time $t = t_0$. The curves $C_{\psi_1(t)_0} \subset (r,z)$ are closed, and the curves $R_{\psi_1(t_0)} \subset (r,z)$ are infinite.

Assume that a surface $\psi_1(r, z, t_0) = C_1$ bounds a compact connected domain D_1. We call the domain D_1 maximal and denote it $\overline{D}_m$ if it is not contained in any bigger compact connected domain D_2 bounded by a surface $\psi_1(r, z, t_0) = C_2$. If such a maximal domain $\overline{D}_m$ intersects the axis of symmetry $r = 0$, then topologically it is a ball $\mathbb{B}^3_m$, which we call a vortex blob because it is invariant with respect to the vorticity field $\nabla\times\mathbf{V}(r, z, t_0)$.

If a maximal compact invariant domain $\overline{D}_m$ does not intersect the axis of symmetry $r = 0$, then it is topologically a 3-dimensional z-axisymmetric ring $\mathbb{B}^2_m(t_0)\times\mathbb{S}^1$, where $\mathbb{B}^2_m(t_0) \subset (r,z)$ topologically is equivalent to a 2-dimensional ball in the poloidal plane (r, z). The boundary of the ring $\mathbb{B}^2_m(t_0)\times\mathbb{S}^1$ is a torus $\mathbb{T}^2 = C_{\psi_1(t_0)}\times\mathbb{S}^1$, where $C_{\psi_1(t_0)} = \partial\mathbb{B}^2_m(t_0)$ is a closed level curve $\psi_1(r, z, t_0) = C_m \neq 0$, $\varphi = 0$. Since the ring $\mathbb{B}^2_m(t_0)\times\mathbb{S}^1$ is invariant with respect to the vorticity field $\nabla\times\mathbf{V}(r, z, t_0)$, we call it a vortex ring. In view of Eq. (2.5), the vortex rings and vortex blobs are also invariant with respect to the velocity field $\mathbf{V}(r, z, t_0)$.

For the z-axisymmetric flows, the vortex rings and vortex blobs are represented by their intersections with the poloidal plane (r, z) where $\varphi = 0$. Below, we study dynamics of the poloidal sections of the vortex rings and blobs for the concrete exact solutions which are obtained from the spheromak solution (2.18) by transformations (2.17).

B. Up-down asymmetric exact solutions

We use the following elementary functions $G_n(u)$ connected with the Bessel functions $J_{n-1/2}(u)$:[26]

$$\begin{aligned}G_2(u) &= \frac{1}{u^2}\left[\cos u - \frac{\sin u}{u}\right] = -\frac{\sqrt{\pi/2}}{u^{3/2}}J_{3/2}(u),\\ G_3(u) &= \frac{\mathrm{d}}{u\mathrm{d}u}G_2(u) = \frac{1}{u^4}\left[(3-u^2)\frac{\sin u}{u} - 3\cos u\right] = \frac{\sqrt{\pi/2}}{u^{5/2}}J_{5/2}(u),\\ G_4(u) &= \frac{\mathrm{d}}{u\mathrm{d}u}G_3(u)\\ &= \frac{1}{u^6}\left[(6u^2-15)\frac{\sin u}{u} - (u^2-15)\cos u\right] = -\frac{\sqrt{\pi/2}}{u^{7/2}}J_{7/2}(u).\end{aligned}\qquad (3.1)$$

All functions $G_n(u)$ are even $[G_n(-u) \equiv G_n(u)]$ and are analytic everywhere and have the nonzero values at $u = 0$: $G_2(0) = -1/3$, $G_3(0) = 1/15$, and $G_4(0) = -1/105$. All functions $G_n(u)$ tend to zero as $|u| \to \infty$ and have infinitely many roots u_j satisfying the equation $G_n(u_j)$

$= 0$. The range of the function $G_2(u)$ is the segment $I^* = [-1/3, \xi_1 \approx 0.028\,72]$.

Applying transformation (2.17) with $a_n = 0$, $m = 2$, $N = 1$, and $b_1 = \alpha^{-2}$ to the stream function $\psi(r, z)$ (2.18), we get a new exact solution to Eqs. (2.3) and (2.4),

$$\psi_2(r,z,t) = \xi r^2 + \frac{1}{\alpha^2}\frac{\partial\psi_1(r,z,t)}{\partial z} = \xi r^2 - \frac{f(t)}{\alpha^2} r^2 \frac{\partial[G_2(\alpha R)]}{\partial z}.$$

Using formula (3.1), $G_3(u) = u^{-1}\mathrm{d}G_2(u)/\mathrm{d}u$, we get

$$\psi_2(r,z,t) = r^2[\xi - zf(t)G_3(\alpha R)]. \tag{3.2}$$

The corresponding velocity field (2.1) has the form

$$\begin{aligned}\mathbf{V}_2(r,z,t) &= rf(t)\left[G_3(\alpha R) + \alpha^2 z^2 G_4(\alpha R)\right]\hat{\mathbf{e}}_r\\ &\quad + \left[2\xi - zf(t)\left[2G_3(\alpha R) + \alpha^2 r^2 G_4(\alpha R)\right]\right]\hat{\mathbf{e}}_z\\ &\quad + \alpha r[\xi - zf(t)G_3(\alpha R)]\hat{\mathbf{e}}_\varphi,\end{aligned} \tag{3.3}$$

where $G_4(u) = u^{-1}\mathrm{d}G_3(u)/\mathrm{d}u$, see formulas in (3.1). Applying results of Sec. II, we get that the fluid velocity field $\mathbf{V}_2(r, z, t)$ (3.3) together with the corresponding pressure $p(r, z, t)$ (2.2) forms a new exact solution to the Navier-Stokes equations (1.1).

Evidently, the stream function (3.2) and the velocity field $\mathbf{V}_2(r, z, t)$ (3.3) are not invariant under the reflection $z \to -z$. Hence, the new exact solution (3.3) is up-down asymmetric.

The velocity field (3.3) for any fixed time t defines the following dynamical system in $\mathbb{R}^3$:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\frac{1}{r}\frac{\partial\psi_2}{\partial z} = rf(t)\left[G_3(\alpha R) + \alpha^2 z^2 G_4(\alpha R)\right], \tag{3.4}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{1}{r}\frac{\partial\psi_2}{\partial r} = 2\xi - zf(t)\left[2G_3(\alpha R) + \alpha^2 r^2 G_4(\alpha R)\right],$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \alpha[\xi - zf(t)G_3(\alpha R)], \tag{3.5}$$

where τ is a new independent parameter.

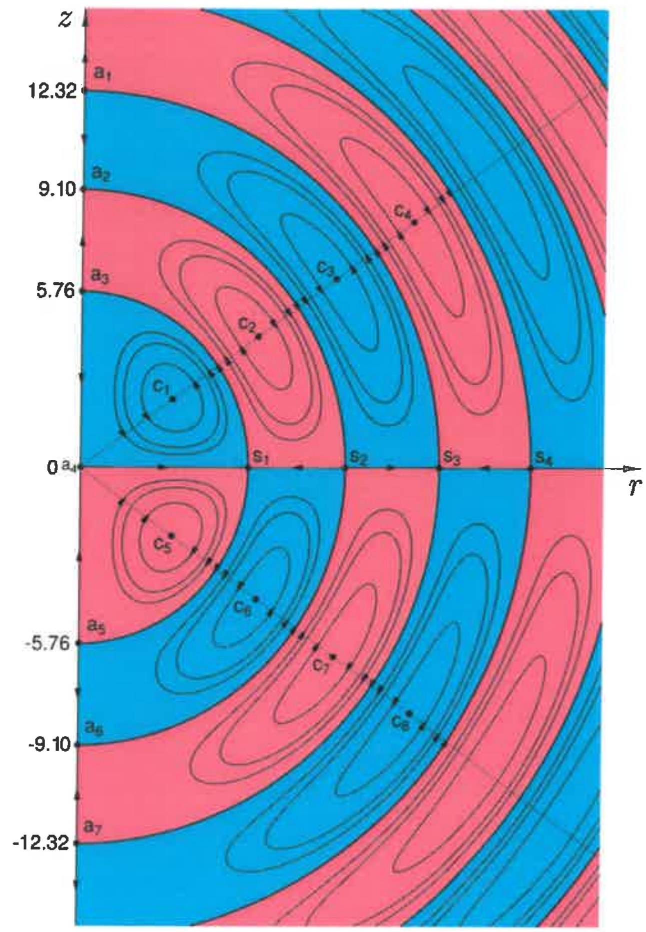

FIG. 1. Poloidal sections $\psi_2(r, z, t)$ = const of stream surfaces for time $t \to -\infty$, $\xi > 0$.

C. Poloidal sections of stream surfaces

Figures 1–10 show (for $\alpha = 1$ and $\xi > 0$) poloidal sections $\psi_2(r, z, t)$ = const of the stream surfaces for the velocity field $\mathbf{V}_2(r, z, t)$ (3.3) for a sequence of increasing moments of time t: $-\infty, t_1 < t_2 < \cdots < t_8, \infty$. The arrows in Figs. 1–10 show the direction of dynamics defined by the system (3.4). The equilibria of the dynamical system (3.4) for a given moment of time t are the points of extrema of the stream function $\psi_2(r, z, t)$ (3.2). The stable equilibria are denoted as c_j; the unstable saddle equilibria are denoted as a_i and s_k. The saddle equilibria s_k belong to the boundaries of vortex rings and a_i belong to the boundaries of vortex blobs. It is evident that no equilibrium points of system (3.4) belong to the plane $z = 0$ or any sphere $R = R_j$ where $G_3(\alpha R_j) = 0$. The vortex rings are shown in Figs. 1–8 in pink color, and vortex blobs are shown in blue.

Remark 3. The stream function (3.2) for $\overline{\xi} = -\xi$ has the form

$$\overline{\psi_2}(r,z,t) = r^2[-\xi - zf(t)G_3(\alpha R)]. \tag{3.6}$$

Formulas (3.2) yield $\overline{\psi_2}(r,z,t) = -\psi_2(r,-z,t)$. Hence, poloidal sections $\overline{\psi_2}(r,z,t) = C_1$ are obtained from the sections $\psi_2(r, z, t) = -C_1$ by the reflection $z \to -z$. Therefore, the figures showing the poloidal sections $\overline{\psi_2}(r,z,t)$ = const for $\overline{\xi} = -\xi < 0$ can be obtained from Figs. 1–10 for $\xi > 0$ by the reflection $z \to -z$.

D. Evolution of vortex blobs and vortex rings in time

Proposition 2. Each vortex blob for the viscous fluid flow (3.3) stays for all times t either in the domain $z > 0$ or in the domain $z < 0$ and never crosses the plane $z = 0$. For all times t, each vortex blob stays inside a spherical shell $R_k < R < R_{k+1}$, where R_k and R_{k+1} are subsequent roots of the equation $G_3(\alpha R) = 0$ and never crosses the sphere $R = R_k$ or $R = R_{k+1}$. Inside each vortex blob, the equation

$$\mathrm{sign}(z)\mathrm{sign}(G_3(\alpha R)) = \mathrm{sign}(\xi) \tag{3.7}$$

holds. Each vortex blob $\overline{D}_m(t_0)$ for any time $t > t_0$ contracts inside itself: $\overline{D}_m(t) \subset \overline{D}_m(t_0)$, until it collapses at some time $t = t_*$ into a point ($r = 0$, $z = z_*$), where t_* and z_* satisfy the equation

$$z_* G_3(\alpha z_*) = \xi/f(t_*), \tag{3.8}$$

provided that $f(t) \to 0$ as $t \to \infty$.

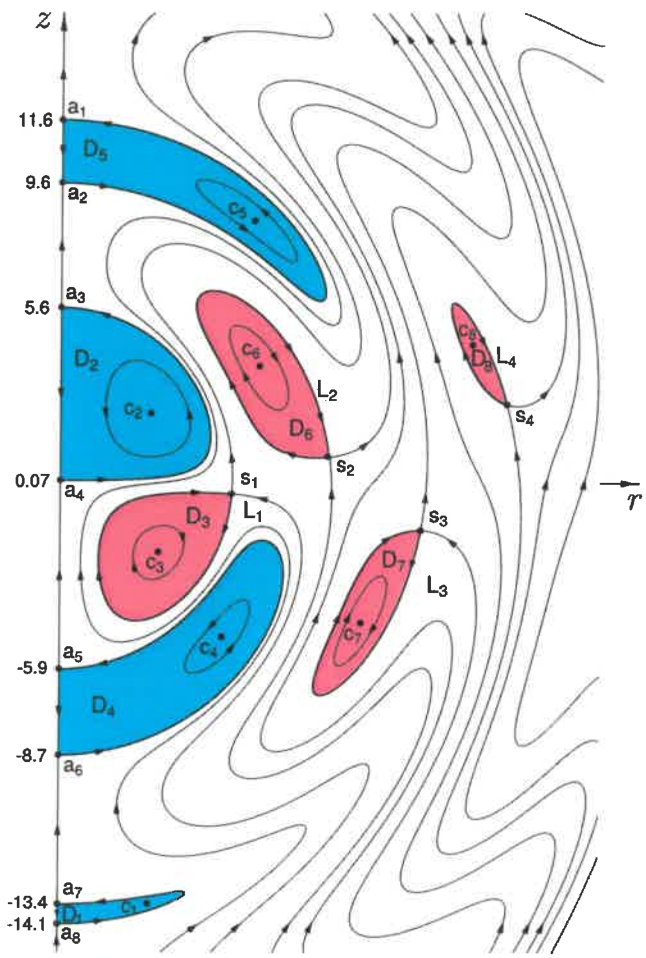

FIG. 2. Poloidal sections $\psi_2(r, z, t_1)$ = const of stream surfaces for time t_1: $f(t_1) = \xi/0.005$.

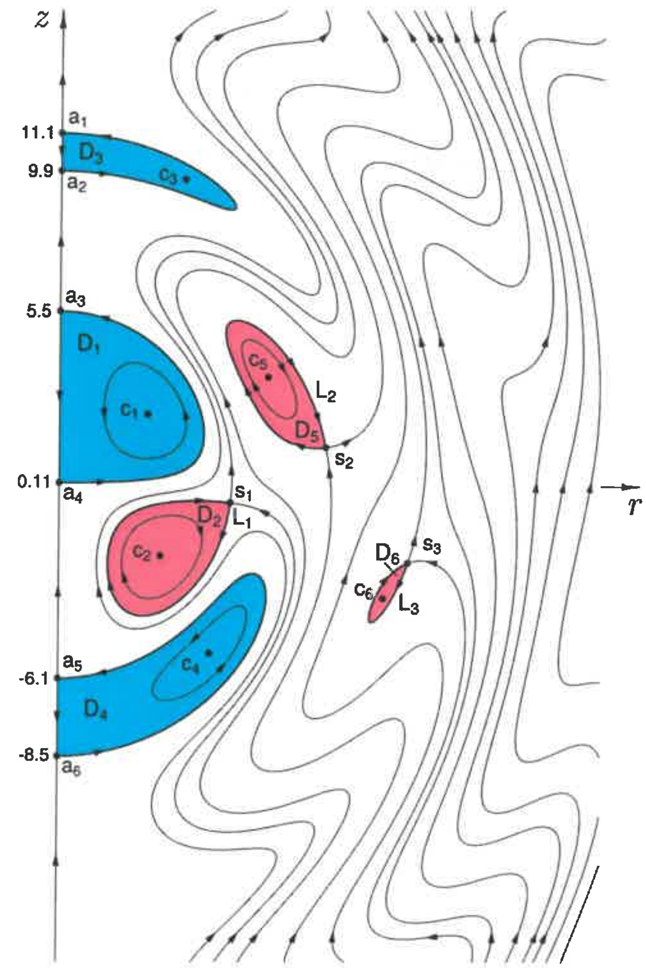

FIG. 3. Poloidal sections $\psi_2(r, z, t_2)$ = const of stream surfaces for time t_2: $f(t_2) = \xi/0.0075$.

Proof. Any vortex blob $\overline{D}_m$ intersects the axis of symmetry $r = 0$. Hence, its boundary also intersects the axis $r = 0$. The boundary at any time $t = t_0$ satisfies the equation $\psi_2(r, z, t_0) = \overline{C}_m$. Since the boundary intersects the axis $r = 0$ and the stream function $\psi_2(r, z, t)$ (3.2) vanishes at $r = 0$, we conclude that $\overline{C}_m = 0$. Hence, the boundary of each vortex blob satisfies the equation $\psi_2(r, z, t) = 0$. This equation in view of (3.2) has the equivalent form

$$zG_3(\alpha R) = \xi/f(t). \tag{3.9}$$

Since the function $f(t) = \exp\left[-\alpha^2 \int_0^t \nu(\tau)d\tau\right] > 0$ and $\xi \neq 0$, we get from Eq. (3.9) that, at the boundary of each vortex blob, $z \neq 0$ and $G_3(\alpha R) \neq 0$ for all times t.

The inequality $z \neq 0$ at the boundary $\partial\overline{D}_m$ of a vortex blob $\overline{D}_m$ yields that the boundary $\partial\overline{D}_m$ lies either in the domain $z > 0$ or in the domain $z < 0$. Hence, all interior points of the blob $\overline{D}_m$ also lie in the domain $z > 0$ or in the domain $z < 0$. Therefore, sign(z) is the same for all points of the vortex blob $\overline{D}_m$ for all times t.

The inequality $G_3(\alpha R) \neq 0$ at the boundary of each vortex blob implies that the boundary $\partial\overline{D}_m$ lies between two spheres $R = R_k$ and $R = R_{k+1}$, where R_k and R_{k+1} are two subsequent roots of the equation $G_3(\alpha R) = 0$. Hence, all interior points of the blob $\overline{D}_m$ also lie between the two spheres $R = R_k$ and $R = R_{k+1}$. Therefore, sign($G_3(\alpha R)$) is the same for all points of the blob $\overline{D}_m$ for all times t.

Equation (3.9) is true at the boundary $\partial\overline{D}_m$ of a vortex blob $\overline{D}_m$ and evidently implies Eq. (3.7) at the boundary. Since both sign(z) and sign($G_3(\alpha R)$) are constant inside the whole vortex blob $\overline{D}_m$, we get that Eq. (3.7) holds for all points of the vortex blob $\overline{D}_m$ for all times t.

The coordinate z of a point (z, R) on the boundary of a vortex blob $\overline{D}_m(t)$ is uniquely defined by Eq. (3.9): $z = \xi/[f(t)G_3(\alpha R)]$. For any fixed t_0 and $R = R_0$, all points of the arc (z, R_0) satisfying inequalities $\xi/[f(t_0)G_3(\alpha R_0)] < |z| \leq R_0$ are interior points of the blob $\overline{D}_m(t_0)$ because no one of them belongs to the boundary $\partial\overline{D}_m(t_0)$. Since the function $f(t)$ (2.14) monotonously decreases when time t is growing, we get that $|z| = |\xi/[f(t)G_3(\alpha R_0)]|$ monotonously increases until at some time t_0^*, the arc collapses into the point $|z| = R_0$. The time t_0^* satisfies the equation $f(t_0^*) = |\xi/[R_0G_3(\alpha R_0)]|$. Therefore, for all times $t > t_0$ and $t \leq t_0^*$, each vortex blob $\overline{D}_m(t)$ is embedded into the blob $\overline{D}_m(t_0)$: $\overline{D}_m(t) \subset \overline{D}_m(t_0)$.

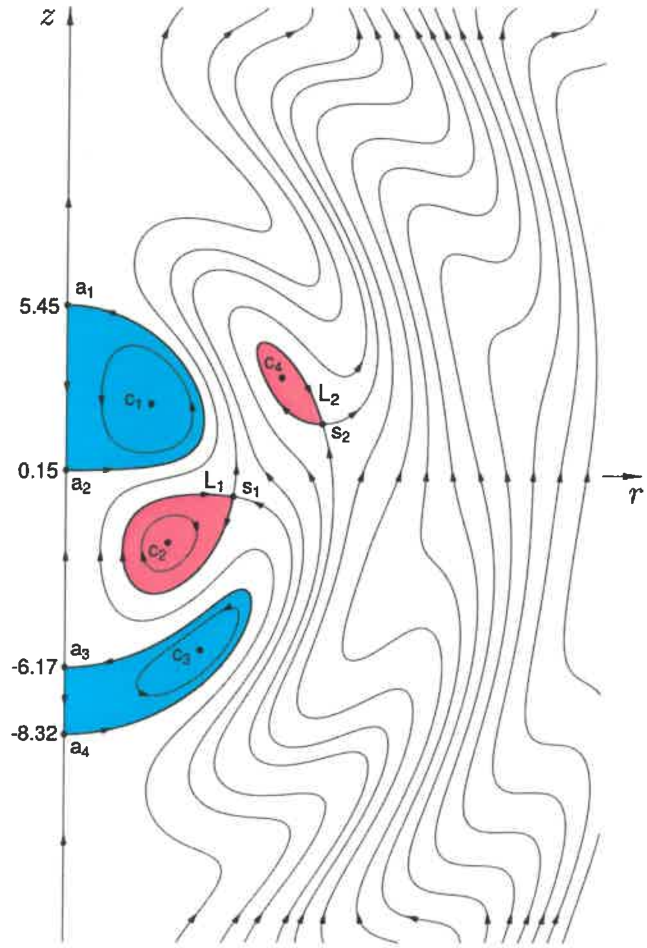

FIG. 4. Poloidal sections $\psi_2(r, z, t_3)$ = const of stream surfaces for time t_3: $f(t_3) = \xi/0.01$.

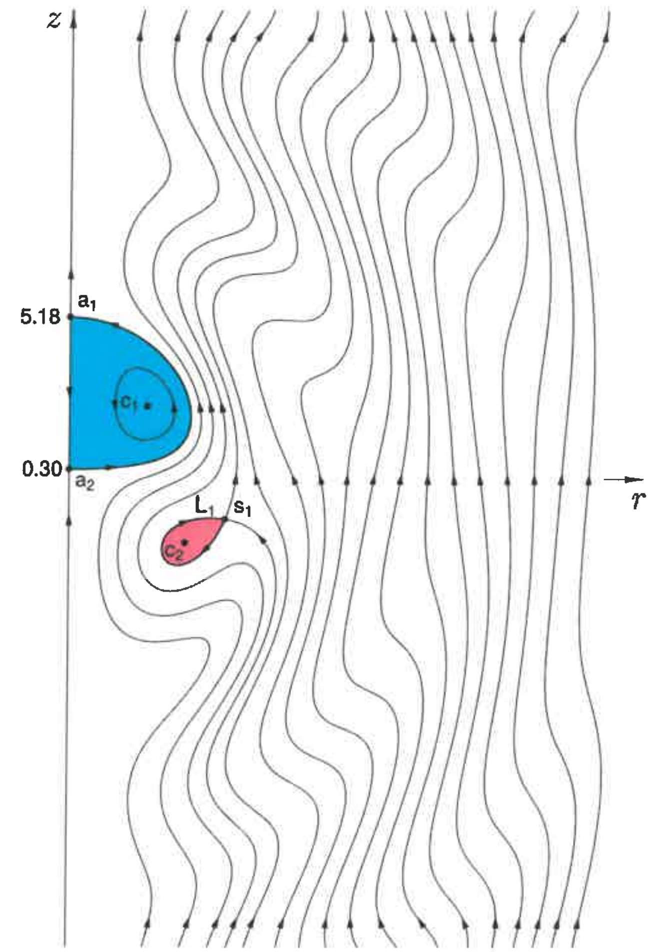

FIG. 5. Poloidal sections $\psi_2(r, z, t_4)$ = const of stream surfaces for time t_4: $f(t_4) = \xi/0.02$.

Assuming that $f(t) \to 0$ as $t \to \infty$, we get that the vector field $\mathbf{V}_2(r, z, t)$ (3.3) tends as $t \to \infty$ to the steady vector field

$$\mathbf{V}(r,z) = 2\xi\hat{\mathbf{e}}_z + \alpha\xi r\hat{\mathbf{e}}_\varphi \tag{3.10}$$

that has no vortex ring or blobs, see Fig. 10. Therefore, each vortex blob for the vector field (3.3) eventually collapses at some time $t = t_*$ into a point ($r = r_* = 0$, $z = z_*$) on the axis of symmetry. Since Eq. (3.9) is true at the boundary of the vortex blob for all times $t \le t_*$ and at the moment t_* of collapse, the radius R is $R = \sqrt{(r^*)^2 + (z^*)^2} = |z^*|$, and we get that Eq. (3.9) at $t = t_*$ yields Eq. (3.8). □

Remark 4. Equation $G_3(\alpha R) = 0$ due to formulas in (3.1) has the equivalent form

$$\tan(\alpha R) = \frac{3\alpha R}{3 - \alpha^2 R^2}. \tag{3.11}$$

It is evident that Eq. (3.11) has infinitely many roots with asymptotics $\alpha R_k \approx (k + 1)\pi$ at $k \to \infty$. The first four roots of Eq. (3.11) are

$$\alpha R_1 \approx 5.7635, \quad \alpha R_2 \approx 9.0950, \quad \alpha R_3 \approx 12.3224, \quad \alpha R_4 \approx 15.5146. \tag{3.12}$$

The vortex blobs are obtained from the pink domains in Figs. 1–8 by rotating them around the axis z, and the vortex rings are obtained by rotating the pink domains. The above results concerning the location of vortex blobs are in a complete agreement with the numerical data shown for $\alpha = 1$ and $\xi > 0$ in Figs. 1–8 and the approximate formulas (3.12) for the roots R_k of the equation $G_3(\alpha R) = 0$. Assuming that $f(t) \to \infty$ as $t \to -\infty$ (for example, if $\nu(t) \ge \nu_0 > 0$), we get from Eq. (3.9) that $zG_3(\alpha R) \to 0$ at the boundary of a vortex blob as $t \to -\infty$. Hence, the boundary of a vortex blob asymptotically consists of parts of spheres $r = R_k$ and $R = R_{k+1}$ where $G_3(\alpha R_k) = G_3(\alpha R_{k+1}) = 0$ and a part of the plane $z = 0$. The limiting poloidal sections of the corresponding domains are shown in Fig. 1 in blue color. The complementary pink parts of Fig. 1 are filled by vortex rings.

Figures 1–10 illustrate dynamics in time t of vortex blobs and vortex rings. Equation (3.9) for the boundary of vortex blobs implies

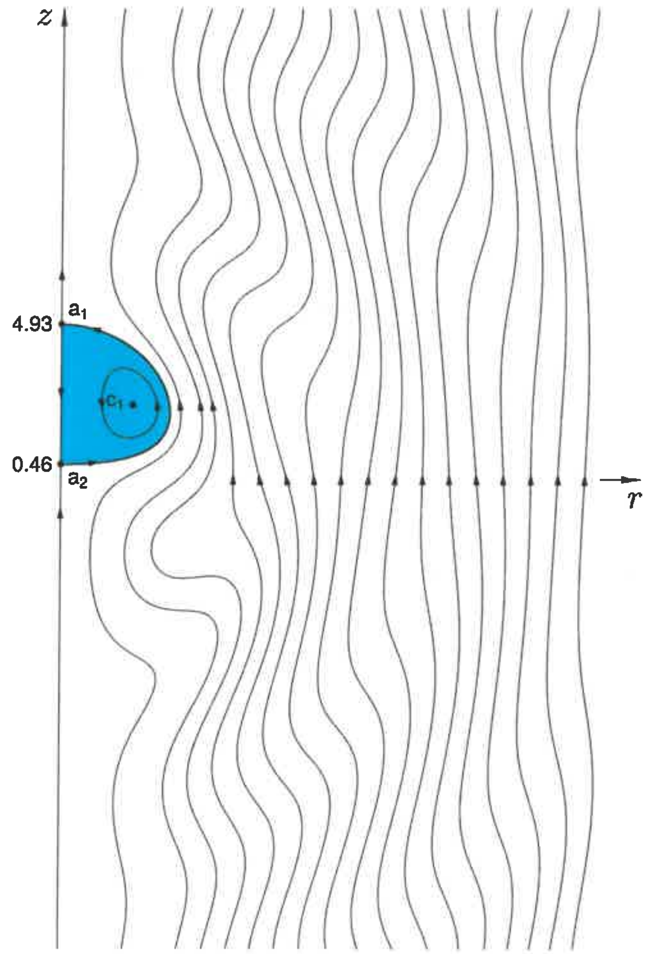

FIG. 6. Poloidal sections $\psi_2(r, z, t_5)$ = const of stream surfaces for time t_5: $f(t_5) = \xi/0.03$.

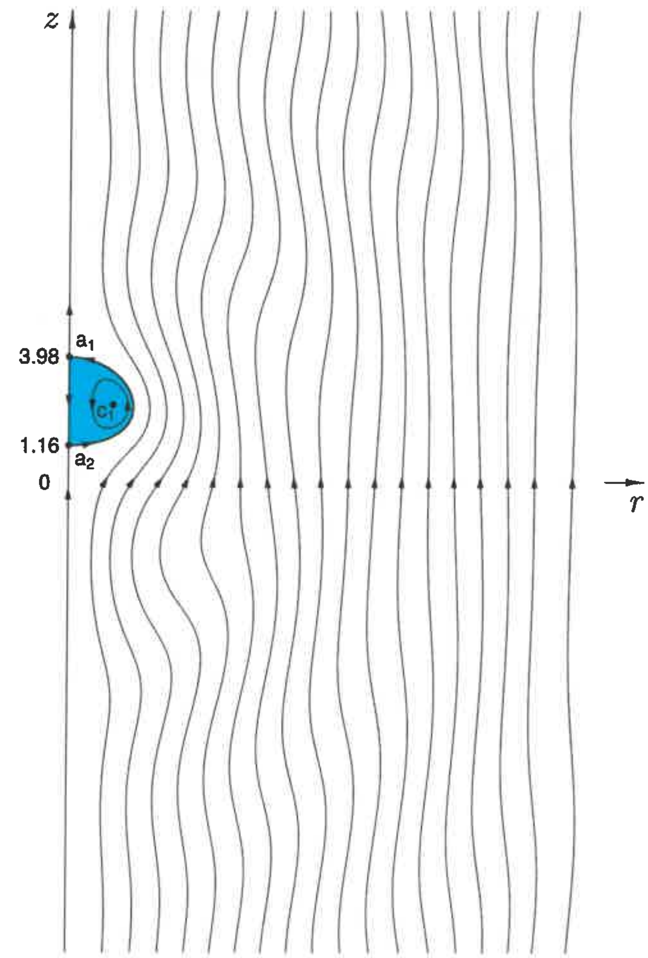

FIG. 7. Poloidal sections $\psi_2(r, z, t_6)$ = const of stream surfaces for time t_6: $f(t_6) = \xi/0.07$.

that, for $\xi \neq 0$, each vortex blob stays for all times t either in the domain $z > 0$ or in the domain $z < 0$ and never crosses the plane $z = 0$. This is visible also from Figs. 2–8. From these figures, it is evident that, for the exact solution (3.3), the vortex rings and vortex blobs collapse and disappear when $t \to \infty$.

The more general stream functions

$$\psi_3(r,z,t) = \xi r^2 + b_1 \frac{\partial \psi_1(r,z,t)}{\partial z} + b_2 \frac{\partial^2 \psi_1(r,z,t)}{\partial z^2}$$
$$= r^2\left[\xi - \alpha^2 f(t)\left[b_1 z G_3(\alpha R) + b_2\left(G_3(\alpha R) + \alpha^2 z^2 G_4(\alpha R)\right)\right]\right] \quad (3.13)$$

are up-down asymmetric if $b_1 \neq 0$. The corresponding exact solutions to the Navier-Stokes equations (1.1) are vector fields $\mathbf{V}_3(r, z, t)$ of form (2.1) and the pressure $p(r, z, t)$ of form (2.2). The solutions are evidently up-down asymmetric if $b_1 \neq 0$.

Using results of Sec. II, one can construct infinitely many exact solutions (2.1)–(2.4) with the stream functions $\psi_m(r, z, t)$ containing partial derivatives $\partial^k \psi_1(r, z, t)/\partial z^k$ for $k = 1, \ldots, m - 1$ and generalizing the stream functions $\psi_2(r, z, t)$ (3.2) and $\psi_3(r, z, t)$ (3.13).

IV. APPLICATIONS TO VISCOUS AND INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

A. Exact axisymmetric solutions with collinear plasma velocity and magnetic field

Let us present some applications of the above results to plasma physics. As known,[27] equations of viscous and incompressible magnetohydrodynamics (MHD) have the form

$$\frac{\partial \mathbf{V}}{\partial t} = -(\nabla \times \mathbf{V}) \times \mathbf{V} + \frac{1}{\rho\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla\left[\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 - \Psi\right] + \nu \nabla^2 \mathbf{V}, \quad (4.1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \nu_m \nabla^2 \mathbf{B}, \quad (4.2)$$

$$\nabla \cdot \mathbf{V} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \quad (4.3)$$

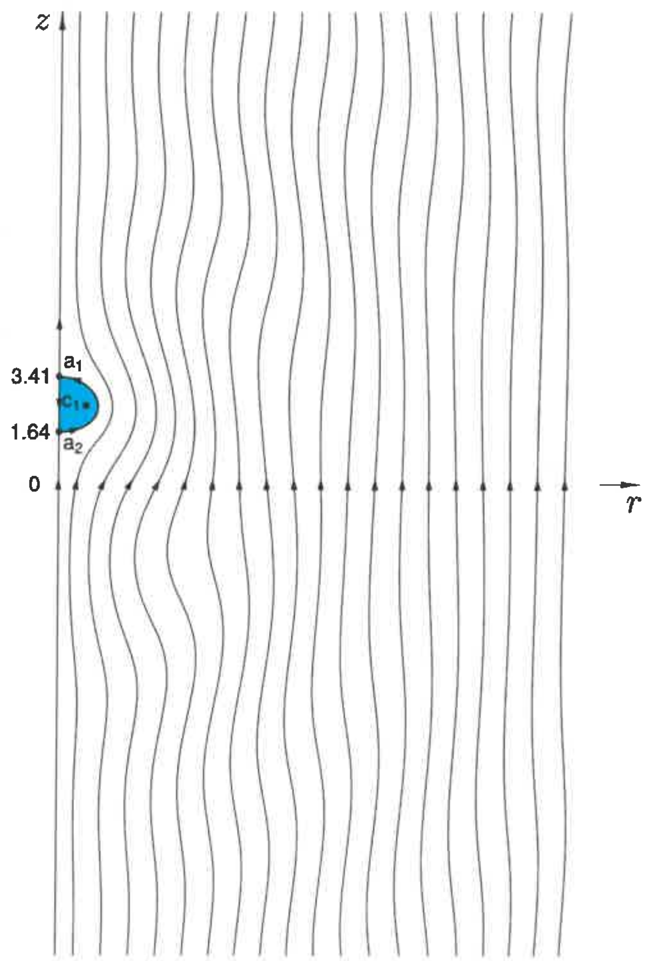

FIG. 8. Poloidal sections $\psi_2(r, z, t_7)$ = const of stream surfaces for time t_7: $f(t_7) = \xi/0.09$.

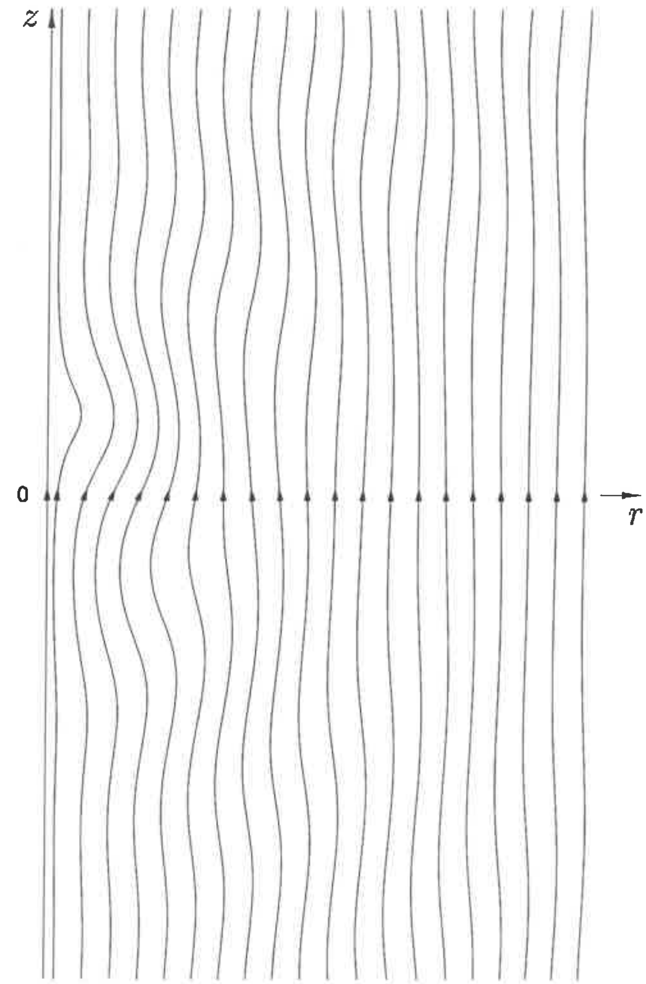

FIG. 9. Poloidal sections $\psi_2(r, z, t_8)$ = const of stream surfaces for time t_8: $f(t_8) = \xi/0.105$.

where **B** is the magnetic field, μ is the magnetic permeability, and $\nu_m(t)$ is the magnetic viscosity; other notations are the same as in Secs. I and II.

First we consider z-axisymmetric solutions to Eqs. (4.1)–(4.3) with the collinear magnetic field and plasma velocity: $\mathbf{B}(r, z, t) = \lambda\mathbf{V}(r, z, t)$, where λ is an arbitrary constant. Assume that plasma velocity $\mathbf{V}(r, z, t)$ has the form (2.1) with the stream function $\psi_1(r, z, t)$ satisfying Eqs. (2.3) and (2.4). In Sec. II, we proved that there exists an infinite-dimensional space of such vector fields. Let us verify that Eqs. (4.1)–(4.3) are satisfied provided that the kinematic viscosity $\nu(t)$ and magnetic viscosity $\nu_m(t)$ coincide: $\nu_m(t) \equiv \nu(t)$. It is evident that Eq. (4.3) holds. Using Eq. (2.7), we find

$$-(\nabla\times\mathbf{V})\times\mathbf{V}+\frac{1}{\rho\mu}(\nabla\times\mathbf{B})\times\mathbf{B}=\nabla\left[\alpha^2\xi\left(1-\frac{\lambda^2}{\rho\mu}\right)\psi_1(r,z,t)\right]. \tag{4.4}$$

Therefore, Eq. (4.1) takes the form

$$\frac{\partial\mathbf{V}}{\partial t}=-\nabla\left[\frac{p}{\rho}+\frac{1}{2}|\mathbf{V}|^2-\Psi-\alpha^2\xi\left(1-\frac{\lambda^2}{\rho\mu}\right)\psi_1(r,z,t)\right]+\nu\nabla^2\mathbf{V}. \tag{4.5}$$

Defining pressure $p(r, z, t)$ by the formula

$$p(r,z,t)=\rho\left[C-\frac{1}{2}|\mathbf{V}(r,z,t)|^2+\Psi(r,z,t)+\alpha^2\xi\left(1-\frac{\lambda^2}{\rho\mu}\right)\psi_1(r,z,t)\right], \tag{4.6}$$

we reduce Eq. (4.5) to the equation

$$\frac{\partial\mathbf{V}}{\partial t}=\nu(t)\nabla^2\mathbf{V}. \tag{4.7}$$

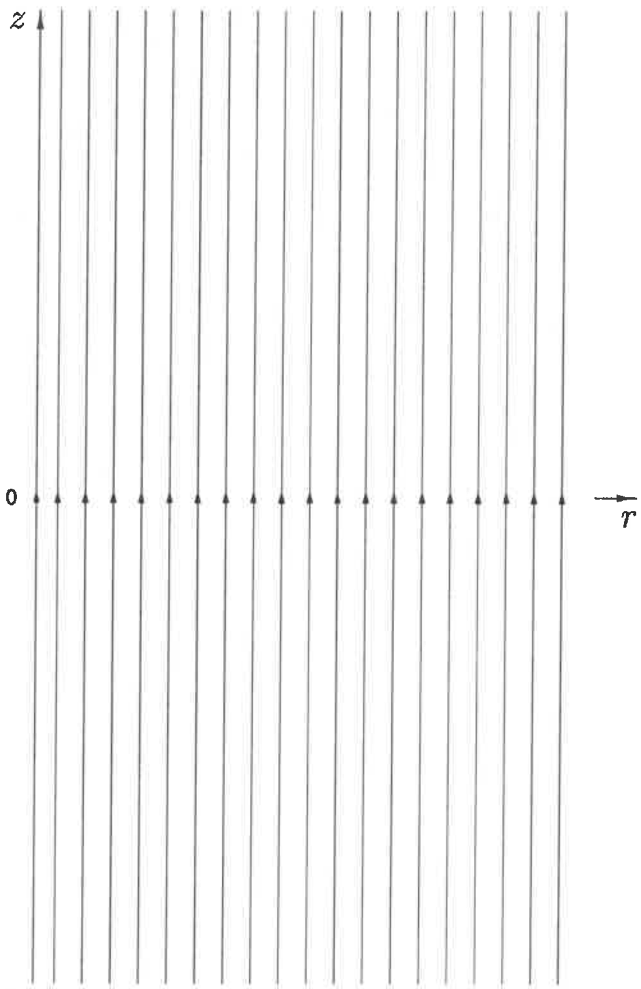

FIG. 10. Poloidal sections $\psi_2(r, z, t)$ = const of stream surfaces for time $t \to \infty$.

The collinearity of vector fields **B** and **V** yields $\mathbf{V} \times \mathbf{B} = 0$. Therefore, Eq. (4.2) is equivalent to

$$\frac{\partial \mathbf{B}}{\partial t} = \nu_m(t)\nabla^2\mathbf{B}. \tag{4.8}$$

The assumptions $\nu(t) = \nu_m(t)$ and $\mathbf{B}(r, z, t) = \lambda\mathbf{V}(r, z, t)$ yield that Eqs. (4.7) and (4.8) are equivalent. Equation (4.7) coincides with Eq. (2.9). We have proved in Proposition 1 that Eq. (2.9) is identically satisfied by the vector field (2.1) with the stream function $\psi_1(r, z, t)$ satisfying Eqs. (2.3) and (2.4).

Therefore, the plasma velocity $\mathbf{V}(r, z, t)$ (2.1), (2.3), and (2.4) and magnetic field $\mathbf{B}(r, z, t) = \lambda\mathbf{V}(r, z, t)$ and pressure $p(r, z, t)$ (4.6) define exact solutions to the viscous and incompressible MHD equations (4.1)–(4.3).

For the constructed field-aligned solutions, magnetic surfaces $\psi_1(r, z, t)$ = const coincide with the vortex surfaces. Let plasma velocity has the exact up-down asymmetric form $\mathbf{V}_2(r, z, t)$ (3.3). Then, dynamics of magnetic rings and blobs which coincide with the vortex rings and blobs is the same as for the exact solution (3.3) studied in Sec. III. As time $t \to \infty$, the plasma velocity $\mathbf{V}(r, z, t)$ and magnetic field $\mathbf{B}(r, z, t)$ tend to the steady fields [see Eq. (3.10)]

$$\mathbf{V}(r,z,t) \longrightarrow 2\xi\hat{\mathbf{e}}_z + \alpha\xi r\hat{\mathbf{e}}_\varphi, \quad \mathbf{B}(r,z,t) \longrightarrow 2\lambda\xi\hat{\mathbf{e}}_z + \alpha\lambda\xi r\hat{\mathbf{e}}_\varphi, \tag{4.9}$$

having constant vorticity $\mathbf{\Omega} = 2\alpha\xi\hat{\mathbf{e}}_z$ and constant electric current $\mathbf{J} = 2\alpha\lambda\xi\hat{\mathbf{e}}_z$. Therefore, during the plasma relaxation, all magnetic rings and blobs collapse and disappear, see Figs. 1–10.

B. Exact solutions with time-dependent plasma velocity and constant electric current

Assume that the kinematic viscosity $\nu(t)$ and magnetic viscosity $\nu_m(t)$ do not coincide and that plasma velocity $\mathbf{V}(r, z, t)$ has the form (2.1) with the stream function $\psi_1(r, z, t)$ satisfying Eqs. (2.3) and (2.4). The space of such vector fields is infinite-dimensional.

Let us verify that Eqs. (4.1)–(4.3) hold provided that the magnetic field $\mathbf{B}(r,z) = \beta r\hat{\mathbf{e}}_\varphi$, where β = const. In the Cartesian coordinates x, y, and z, the magnetic field $\mathbf{B}(r, z)$ is $\mathbf{B}(x,y,z) = -\beta y\hat{\mathbf{e}}_x + \beta x\hat{\mathbf{e}}_y$. It is evident that

$$\nabla \times \mathbf{B}(r,z) = 2\beta\hat{\mathbf{e}}_z, \quad \nabla \cdot \mathbf{B}(r,z) = 0, \quad \nabla^2\mathbf{B}(r,z) = 0. \tag{4.10}$$

The first Eq. (4.10) yields that electric current $\mathbf{J}(r, z) = \nabla \times \mathbf{B}(r, z)$ is constant. From this equation, we find

$$(\nabla \times \mathbf{B}(r,z)) \times \mathbf{B}(r,z) = 2\beta\hat{\mathbf{e}}_z \times (\beta r\hat{\mathbf{e}}_\varphi) = -2\beta^2 r\hat{\mathbf{e}}_r = \nabla(-\beta^2 r^2). \tag{4.11}$$

From Eqs. (2.7) and (4.11), we find

$$-(\nabla \times \mathbf{V}) \times \mathbf{V} + \frac{1}{\rho\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla\left[\alpha^2\xi\psi_1(r,z,t) - \frac{\beta^2 r^2}{\rho\mu}\right]. \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.1), we get

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla\left[\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 - \Psi - \alpha^2\xi\psi_1(r,z,t) + \frac{\beta^2 r^2}{\rho\mu}\right] + \nu\nabla^2\mathbf{V}. \tag{4.13}$$

Let us define pressure $p(r, z, t)$ by the formula

$$p(r,z,t) = \rho\left[C - \frac{1}{2}|\mathbf{V}(r,z,t)|^2 + \Psi(r,z,t) + \alpha^2\xi\psi_1(r,z,t) - \frac{\beta^2 r^2}{\rho\mu}\right]. \tag{4.14}$$

Substituting Eq. (4.14) into (4.13), we reduce it to Eq. (4.7) that is equivalent to Eq. (2.9) that due to Proposition 1 is identically satisfied by the vector field $\mathbf{V}(r, z, t)$ (2.1) with the stream function $\psi_1(r, z, t)$ obeying Eqs. (2.3) and (2.4). Therefore, Eq. (4.1) holds.

Using formula (2.1), we find

$$\begin{aligned}\mathbf{V} \times \mathbf{B} &= \left[-\frac{1}{r}\frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha\psi_1}{r}\hat{\mathbf{e}}_\varphi\right] \times (\beta r\hat{\mathbf{e}}_\varphi) \\ &= -\beta\left[\frac{\partial\psi_1}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial\psi_1}{\partial z}\hat{\mathbf{e}}_z\right] = -\beta\nabla\psi_1(r,z,t).\end{aligned} \tag{4.15}$$

Equation (4.15) yields

$$\nabla \times (\mathbf{V} \times \mathbf{B}) = -\beta\nabla \times \nabla\psi_1(r,z,t) = 0. \tag{4.16}$$

Equation (4.16) and equations $\partial \mathbf{B}(r, z)/\partial t = 0$ and $\nabla^2 \mathbf{B}(r, z) = 0$ imply that Eq. (4.2) is identically satisfied.

Therefore, the plasma velocity $\mathbf{V}(r, z, t)$ (2.1), (2.3), and (2.4) and magnetic field $\mathbf{B}(r,z) = \beta r \hat{\mathbf{e}}_\varphi$ with constant electric current $\mathbf{J} = 2\beta \hat{\mathbf{e}}_z$ together with the pressure $p(r,z,t)$ (4.14) form exact solutions to the viscous MHD equations (4.1)–(4.3).

Taking plasma velocity in the up-down asymmetric form $\mathbf{V}_2(r, z, t)$ (3.3), we get exact solution describing relaxation of plasma with collapsing vortex rings and blobs in the presence of the magnetic field $\mathbf{B}(r,z) = \beta r \hat{\mathbf{e}}_\varphi$ with constant electric current $\mathbf{J} = 2\beta \hat{\mathbf{e}}_z$.

C. Exact solutions with constant plasma vorticity and time-dependent magnetic field

Suppose that the kinematic viscosity $\nu(t)$ and magnetic viscosity $\nu_m(t)$ are arbitrary and that magnetic field $\mathbf{B}(r, z, t)$ is defined by the formula analogous to (2.1),

$$\mathbf{B}(r,z,t) = -\frac{1}{r}\frac{\partial \psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha \psi_1}{r}\hat{\mathbf{e}}_\varphi, \tag{4.17}$$

where the magnetic function $\psi_1(r, z, t)$ satisfies Eqs. (2.3) and (2.4) with magnetic viscosity $\nu_m(t)$ instead of $\nu(t)$. Analogously to Eq. (2.7), we find

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla\left[-\alpha^2 \xi \psi_1(r,z,t)\right]. \tag{4.18}$$

Let us verify that Eqs. (4.1)–(4.3) are satisfied provided that plasma velocity has the form $\mathbf{V}(r,z) = \gamma r \hat{\mathbf{e}}_\varphi$, where $\gamma = \text{const.}$ Evidently, we have

$$\nabla \times \mathbf{V}(r,z) = 2\gamma \hat{\mathbf{e}}_z, \quad \nabla \cdot \mathbf{V}(r,z) = 0, \quad \nabla^2 \mathbf{V}(r,z) = 0. \tag{4.19}$$

Hence, vorticity $\mathbf{\Omega} = \nabla \times \mathbf{V} = 2\gamma \hat{\mathbf{e}}_z$ is constant in the space and

$$-(\nabla \times \mathbf{V}) \times \mathbf{V} = -2\gamma \hat{\mathbf{e}}_z \times (\gamma r \hat{\mathbf{e}}_\varphi) = 2\gamma^2 r \hat{\mathbf{e}}_r = \nabla(\gamma^2 r^2). \tag{4.20}$$

Equations (4.18) and (4.20) imply

$$-(\nabla \times \mathbf{V}) \times \mathbf{V} + \frac{1}{\rho\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla\left[\gamma^2 r^2 - \frac{\alpha^2 \xi}{\rho\mu}\psi_1(r,z,t)\right]. \tag{4.21}$$

Substituting Eq. (4.21) into (4.1), we reduce it to the equation

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla\left[\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 - \Psi - \gamma^2 r^2 + \frac{\alpha^2 \xi}{\rho\mu}\psi_1(r,z,t)\right] + \nu \nabla^2 \mathbf{V}. \tag{4.22}$$

Let us assume that the pressure $p(r, z, t)$ has the form

$$p(r,z,t) = \rho\left[C - \frac{1}{2}|\mathbf{V}(r,z,t)|^2 + \Psi(r,z,t) + \gamma^2 r^2 - \frac{\alpha^2 \xi}{\rho\mu}\psi_1(r,z,t)\right]. \tag{4.23}$$

Equation (4.22) with the pressure (4.23) is identically satisfied because $\partial \mathbf{V}(r, z)/\partial t = 0$ and $\nabla^2 \mathbf{V}(r, z) = 0$. Hence, Eq. (4.1) holds.

Using formula (4.17), we find

$$\mathbf{V}(r,z) \times \mathbf{B}(r,z,t) = \gamma r \hat{\mathbf{e}}_\varphi \times \left[-\frac{1}{r}\frac{\partial \psi_1}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \psi_1}{\partial r}\hat{\mathbf{e}}_z + \frac{\alpha \psi_1}{r}\hat{\mathbf{e}}_\varphi\right]$$
$$= \gamma\left[\frac{\partial \psi_1}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial \psi_1}{\partial z}\hat{\mathbf{e}}_z\right] = \gamma \nabla \psi_1(r,z,t).$$

Hence, we get $\nabla \times (\mathbf{V} \times \mathbf{B}) = \gamma \nabla \times \nabla \psi_1(r, z, t) = 0$. Therefore, Eq. (4.2) reduces to Eq. (4.8). The latter equation is identically satisfied because magnetic field $\mathbf{B}(r, z, t)$ (4.17) has the same form as $\mathbf{V}(r, z, t)$ (2.1), (2.3), and (2.4), and hence, Eq. (4.8) coincides with Eq. (2.9), validity of which was proved in Proposition 1.

Therefore, the plasma velocity $\mathbf{V}(r,z) = \gamma r \hat{\mathbf{e}}_\varphi$ having constant vorticity $\mathbf{\Omega} = 2\gamma \hat{\mathbf{e}}_z$ and magnetic field $\mathbf{B}(r, z, t)$ (4.17) with the magnetic function $\psi_1(r,z,t)$ satisfying Eqs. (2.3) and (2.4) together with the pressure $p(r, z, t)$ (4.23) form exact solutions to the viscous and incompressible MHD equations (4.1)–(4.3).

Let the magnetic field $\mathbf{B}(r, z, t)$ (4.17) has the exact up-down asymmetric form (3.3). Figures 1–10 of Subsection III D describe the evolution of magnetic rings and magnetic blobs in time t. For the exact solution, the plasma relaxes as $t \to \infty$ to the state

$$\mathbf{B}(r,z,t) \longrightarrow 2\xi \hat{\mathbf{e}}_z + \alpha \xi r \hat{\mathbf{e}}_\varphi, \qquad \mathbf{V}(r,z) = \gamma r \hat{\mathbf{e}}_\varphi$$

having constant electric current $\mathbf{J} = 2\alpha\xi \hat{\mathbf{e}}_z$. As $t \to \infty$, all magnetic rings and blobs collapse and disappear (see Figs. 1–10). The plasma vorticity remains constant: $\mathbf{\Omega} = 2\gamma \hat{\mathbf{e}}_z$.

V. CONCLUSION

We presented an infinite-dimensional space of new exact time-dependent axisymmetric solutions (2.1)–(2.4) to the Navier-Stokes equations (1.1). The solutions are analytic in the whole space $\mathbb{R}^3$ and exist for all times t. The vorticity field and velocity field for the derived fluid flows are connected by Eq. (2.7): $(\nabla \times \mathbf{V}) \times \mathbf{V} = -\alpha^2 \xi \nabla \psi_1$, and therefore are not collinear.

Applying transformations (2.17) with $a_n = 0$, $z_{k,n} = 0$, and $N = 1$ to the stream function $\psi_1(r, z, t) = r^2[\xi - f(t)G_2(\alpha R)]$ (2.19), we get an infinite sequence of stream functions,

$$\psi_m(r,z,t) = \xi r^2 + \sum_{n=1}^{m-1} b_n \frac{\partial^n \psi_1(r,z,t)}{\partial z^n}$$
$$= r^2\left[\xi - f(t)\sum_{n=1}^{m-1} b_n \frac{\partial^n G_2(\alpha R)}{\partial z^n}\right], \tag{5.1}$$

that define by formula (2.1) exact solutions $\mathbf{V}_m(r, z, t)$ to the Navier-Stokes equations; here, $m = 2, 3, \ldots$. The stream function $\psi_m(r, z, t)$ (5.1) for $m = 2$ and $b_1 = 1/\alpha^2$ has the form (3.2) the following form: $\psi_2(r,z,t) = r^2[\xi - zf(t)G_3(\alpha R)]$, where the function $G_3(u)$ is presented in (3.1). The stream function $\psi_2(r, z, t)$ and the corresponding vector field $\mathbf{V}_2(r, z, t)$ (2.1) and (3.3) are up-down asymmetric. The stream functions $\psi_m(r, z, t)$ (5.1) and the corresponding vector fields (2.1) are up-down asymmetric if at least one coefficient $b_n \neq 0$ for $n = 2k + 1$.

For the exact up-down asymmetric fluid flows (3.3), we studied the dynamics of vortex rings and vortex blobs. Figures 1–10 describe the poloidal sections of the stream surfaces $\psi_2(r, z, t_k) = \text{const}$ for ten different moments of time t: $-\infty, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, +\infty$ for $\xi > 0$. The figures for the exact solution (3.6) with the parameter $\bar{\xi} = -\xi$ follow from Figs. 1–10 after the reflection $z \to -z$.

The derived exact viscous flows for $f(t) = \exp\left[-\alpha^2 \int_0^t \nu(\tau)d\tau\right] \to 0$ as $t \to \infty$ tend to the steady flow $\mathbf{V}(r,z) = \alpha\xi r \hat{\mathbf{e}}_\varphi + 2\xi \hat{\mathbf{e}}_z$ that has a constant vorticity $\nabla \times \mathbf{V}(r,z) = 2\alpha\xi \hat{\mathbf{e}}_z$ and therefore has no vortex rings and blobs. Hence, for the constructed exact solutions to the

Navier-Stokes equations, the vortex rings and vortex blobs collapse and disappear as $t \to \infty$.

The methods developed for the Navier-Stokes equations in Secs. II and III are applied in Sec. IV to the equations of viscous and incompressible magnetohydrodynamics. Three infinite-dimensional families of exact solutions are obtained which describe three different regimes of plasma relaxation.

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[28] Collisions of symmetrically positioned vortex rings in a viscous incompressible fluid were studied by numerical methods in paper.[10] Experiments on the dynamics of vortex rings in a viscous incompressible fluid were reported in Ref. 11.

[29] A comprehensive analysis of Hicks' papers[16,17] is presented in Sec. 9 "Comments on Hicks' Papers" (pp. 178–179) and footnotes 3 and 4 (p. 166) of our paper.[18]