Rapid Communication

Oleg Bogoyavlenskij*

Invariants of the Axisymmetric Flows of an Inviscid Gas and Fluid with Variable Density

Abstract: Material conservation laws and integral invariants are constructed for the axisymmetric flows of an inviscid compressible gas and an ideal incompressible fluid with variable density $\rho(\mathbf{x}, t)$. The functional independence of the new invariants from helicity is proven.

Keywords: Helicity; Material Conservation Laws; Vortex Blobs; Vortex Rings.

1 Introduction

It is well known that two-dimensional, ideal, incompressible fluid mechanics admits infinitely many integral invariants [1–3] that are based on material conservation laws. These laws are constant along the incompressible fluid streaklines and therefore are also called Lagrangian invariants [4]. For the axisymmetric flows of an ideal incompressible fluid with constant density ρ, the first material conservation law was discovered by Batchelor in 1967 [1, p. 544]. In 2013, Kelbin et al. [5] presented the material conservation laws depending on an arbitrary function $F(\zeta)$ for the helically symmetric and axisymmetric flows of an ideal incompressible fluid with a constant density ρ.

In Section 2 we derive for the axisymmetric flows of an incompressible inviscid fluid with variable density $\rho(\mathbf{x}, t)$ the material conservation laws containing arbitrary differentiable functions $G(x, y)$ of two variables x and y. This result is a generalisation for the case of the variable-density $\rho(\mathbf{x}, t)$ of the results by Batchelor [1] and by Kelbin et al. [5] for flows with a constant density ρ. The material conservation laws presented in [5] depend on function $F(\zeta)$ of one variable ζ. Our method is different from the methods employed in [1] and [5, 6] and uses only the rotational invariance of the pressure $p(\mathbf{x}, t)$.

In Section 2 we study also inviscid compressible gas flows (which were not considered in [5, 6]) with an arbitrary equation of state. We derive infinitely many material conservation laws for the axisymmetric flows of an inviscid compressible gas.

In Section 3 we derive the integral invariants for the axisymmetric flows of the inviscid gas and fluid, which are functions of a real parameter μ.

In Section 4 we define the maximum vortex rings and blobs for the axysimmetric flows of the inviscid gas and fluid and prove that the former are frozen into the gas and fluid flows. We define their corresponding functional and integral invariants.

In Section 5 we prove that the new invariants are functionally independent of the helicity. Serre proved that "any conservation law $\int_M \phi(\mathbf{u}(x, t), \operatorname{grad} \mathbf{u}(x, t))\mathrm{d}x = \text{constant}$ of the three-dimensional Euler equations of the incompressible perfect fluid is a linear combination of momenta, energy, and helicity. Therefore, there are no other invariants of the first order (i.e., involving $\mathbf{u}$ and its first spatial derivatives) than those which are already known" [7, p. 105]. We introduce in this article new invariants *of the first order* that do not belong to the class of conservation laws studied by Serre because they explicitly depend on the spatial variables x_1, x_2, x_3 and therefore are functionally independent of the momenta, kinetic energy, and helicity.

As is known [8], helicity is connected with vortex knots and vortex links. Vortex knots for the axisymmetric fluid equilibria were investigated by Moffatt [8] and by Bogoyavlenskij [9–11], see also the Corrigendum [12].

Helicity was studied by Kudryavtseva [13, 14] and by Enciso et al. [15]. The authors stated that "Helicity is the only integral invariant of volume-preserving transformations" [15]. This statement is the title of the papers by Enciso et al. [15] and Kudryavtseva [14]. However, in [5], in this article, and in [16], new integral invariants are derived that are different from helicity. Nevertheless, this does not contradict the statements of Kudryavtseva and Enciso et al. because these authors studied invariants that are integrals of certain densities "over compact three-dimensional manifold M without boundary" [15, p. 2035]. The invariants constructed in [5], in this article, and in [16] depend explicitly on the Cartesian coordinates x_1, x_2, x_3 in the Euclidean space $\mathbb{R}^3$ and hence cannot be projected

*Corresponding author: Oleg Bogoyavlenskij, Department of Mathematics, Queen's University, Kingston, Canada, E-mail: Bogoyavl@mast.queensu.ca

onto any "compact three-dimensional manifold M without boundary".

We study Euler's equations of the ideal incompressible fluid mechanics

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \mathbf{f},$$

$$\nabla \cdot \mathbf{V} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{V} = 0, \tag{1}$$

and Euler's equations of the inviscid compressible gas dynamics [1]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \mathbf{f}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{3}$$

where $\mathbf{V}(\mathbf{x}, t)$ is the fluid or gas velocity, $p(\mathbf{x}, t)$ is the pressure, $\rho(\mathbf{x}, t)$ is the mass density, $\mathbf{f}(\mathbf{x}, t)$ is the Newtonian gravitational force per unit mass, and ∇ is the nabla operator. The Euler equations (2) and (3) can be completed by an equation for the entropy density $s(\mathbf{x}, t)$ and the equation of state $p = p(\rho, s)$. For the study of invariants of the inviscid gas flows, we use only (2) and (3). Therefore, the results of this article are applicable to the inviscid gas dynamics with any arbitrary equation of state.

2 Material Conservation Laws for Axisymmetric Flows

Any fluid flow that is invariant under rotations around an axis with direction $\mathbf{A}$ evidently becomes a z-axisymmetric flow in Cartesian coordinates with the z-axis in the direction of the vector $\mathbf{A}$.

In the cylindrical coordinates $r = \sqrt{x^2 + y^2}$, z, and $\varphi = \arctan(y/x)$, the z-axisymmetric fluid or gas velocity $\mathbf{V}(r, z, t)$ has the following standard form:

$$\mathbf{V}(r, z, t) = \tilde{u}(r, z, t)\hat{\mathbf{e}}_r + \tilde{v}(r, z, t)\hat{\mathbf{e}}_z + w(r, z, t)\hat{\mathbf{e}}_\varphi, \tag{4}$$

where $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_z, \hat{\mathbf{e}}_\varphi$ are vectors of unit length in the directions of coordinates r, z, φ.

Remark 1: Batchelor had presented in [1, p. 544] for the ideal incompressible fluid with constant density ρ the equation "D(rw)/D$t = 0$ $(7 \cdot 5 \cdot 7)$ representing the constancy of the circulation round a material curve in the form of a circle centered on the axis of symmetry and normal to it". No other consequences of equation D(rw)/D$t = 0$ were derived. Batchelor's equation D(rw)/D$t = 0$ [1] of

1967 was the first material conservation law for the axisymmetric flows of the ideal incompressible fluid with constant density ρ. Batchelor did not write in [1] that, for any differentiable function $F(\zeta)$, there exists the composed material conservation law

$$\frac{\mathrm{D}F(rw)}{\mathrm{D}t} = \frac{\mathrm{d}F(rw)}{\mathrm{d}(rw)}\frac{\mathrm{D}(rw)}{\mathrm{D}t} = 0,$$

probably because for him this was self-evident.

We assume that $\mathbf{f}(\mathbf{x}, t)$ is the gravitational force that is z-axisymmetric and satisfies equation $(\hat{\mathbf{e}}_z \times \mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, t) \equiv 0$, which means $\mathbf{f}(\mathbf{x}, t)$ is pure poloidal.

(a) *Let* $\mathbf{V}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ *describe an inviscid incompressible fluid flow with a variable density $\rho(\mathbf{x}, t)$ and a z-rotationally invariant pressure $p(\mathbf{x}, t) = p(r, z, t)$. Then for any differentiable function $G(x, y)$ of two variables x and y, the composed function*

$$G(\rho(\mathbf{x}, t), M(\mathbf{x}, t)), \qquad M(\mathbf{x}, t) = \left[\mathbf{x} \times \mathbf{V}(\mathbf{x}, t)\right] \cdot \hat{\mathbf{e}}_z \tag{5}$$

is a material conservation law of the fluid flow, which means it is constant along the fluid streaklines.

Indeed, the fluid and gas streaklines satisfy the system of equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{V}(\mathbf{x}, t). \tag{6}$$

Let us prove first that the material derivative D$M(\mathbf{x}, t)$/Dt vanishes. Our proof is equally applicable for compressible gas flows because we do not use here the incompressibility condition. Differentiating the function $M(\mathbf{x}, t)$ with respect to system (6), we get

$$\frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = \left[\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times \mathbf{V}(\mathbf{x}, t) + \mathbf{x} \times \frac{\mathrm{D}\mathbf{V}(\mathbf{x}, t)}{\mathrm{D}t}\right] \cdot \hat{\mathbf{e}}_z$$

$$= \left[\mathbf{V} \times \mathbf{V} + \mathbf{x} \times \left(\frac{\partial \mathbf{V}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathbf{V}}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)\right] \cdot \hat{\mathbf{e}}_z$$

$$= \left[\mathbf{x} \times \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right)\right] \cdot \hat{\mathbf{e}}_z.$$

Substituting here Euler's equation (2), we find

$$\frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = \left[-\frac{1}{\rho}(\mathbf{x} \times \nabla p) + (\mathbf{x} \times \mathbf{f})\right] \cdot \hat{\mathbf{e}}_z. \tag{7}$$

Applying the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$ to (7) and using the equation $(\hat{\mathbf{e}}_z \times \mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, t) \equiv 0$, we transform (7) into

$$\frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = -\frac{1}{\rho}\left[(\hat{\mathbf{e}}_z \times \mathbf{x})\right] \cdot \nabla p = -\frac{1}{\rho}\left[\mathbf{U}(\mathbf{x})\right] \cdot \nabla p, \tag{8}$$

where $\mathbf{U}(\mathbf{x}) = \hat{\mathbf{e}}_z \times \mathbf{x}$. The vector field $\mathbf{U}(\mathbf{x})$ generates the one-parametric group of rotations around the z-axis with the unit angular velocity. Therefore, $\mathbf{U}(\mathbf{x}) \cdot \nabla p = \nabla_{\mathbf{U}(\mathbf{x})} p = \partial p / \partial \varphi$. Substituting this into (8) and using the z-rotational invariance of the pressure $p = p(r, z, t)$, we get

$$\frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = -\frac{1}{\rho} \frac{\partial p}{\partial \varphi} = 0. \qquad (9)$$

Let us use now the incompressibility condition. For incompressible fluid flow, we see from the third of the set (1) that $\mathrm{D}\rho(\mathbf{x}, t)/\mathrm{D}t = 0$, which means that the fluid density $\rho(\mathbf{x}, t)$ is constant along the fluid streaklines. Equation (9) states that $\mathrm{D}M(\mathbf{x}, t)/\mathrm{D}t = 0$. Therefore, the material derivative of the composed function $G(\rho(\mathbf{x}, t), M(\mathbf{x}, t))$ is

$$\frac{\mathrm{D}G(\rho, M)}{\mathrm{D}t} = \frac{\partial G(\rho, M)}{\partial \rho} \frac{\mathrm{D}\rho(\mathbf{x}, t)}{\mathrm{D}t} + \frac{\partial G(\rho, M)}{\partial M} \frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = 0. \qquad (10)$$

Hence all functions $G(\rho(\mathbf{x}, t), M(\mathbf{x}, t))$ are the material conservation laws.

Remark 2: The statement (a) is applicable for any z-axisymmetric solutions to Euler's equations (1). However, in the proof of the statement (a), we use only the z-rotational invariance of the pressure $p = p(r, z, t)$.

Remark 3: We proved in statement (a) that the function $M(\mathbf{x}, t)$ (5) is constant along the streaklines of the inviscid gas or fluid. However, it is not constant along the streamlines that are defined by the equation $\mathrm{d}\mathbf{x}/\mathrm{d}\tau = \mathbf{V}(\mathbf{x}, t)$, where τ is a new parameter and $t = \mathrm{const}$. Streaklines and streamlines do not coincide for gas or fluid flows that depend on time t.

(b) *For any differentiable functions $F(x)$ and any flow of an inviscid compressible gas with a z-rotationally invariant pressure $p(\mathbf{x}, t) = p(r, z, t)$, the composed functions*

$$F(M(\mathbf{x}, t)), \qquad M(\mathbf{x}, t) = [\mathbf{x} \times \mathbf{V}(\mathbf{x}, t)] \cdot \hat{\mathbf{e}}_z \qquad (11)$$

are material conservation laws, which means $F(M(\mathbf{x}, t))$ are constant along the compressible gas streaklines.

Indeed, we proved in statement (a) that $\mathrm{D}M(\mathbf{x}, t)/\mathrm{D}t = 0$, see (9). Evidently, for the composed functions $F(M(\mathbf{x}, t))$, we have

$$\frac{\mathrm{D}F(M(\mathbf{x}, t))}{\mathrm{D}t} = \frac{\mathrm{d}F(M)}{\mathrm{d}M} \frac{\mathrm{D}M(\mathbf{x}, t)}{\mathrm{D}t} = 0. \qquad (12)$$

Hence the composed functions $F(M(\mathbf{x}, t))$ are the material conservation laws.

Remark 4: Statements (a) and (b) imply that the diffeomorphisms defined by the axisymmetric fluid and gas flows preserve the values of function $M(\mathbf{x}, t)$ (5). Hence all associated geometric objects such as the range of function $M(\mathbf{x}, t)$, its level sets $M(\mathbf{x}, t) = \mu$ for any constant μ, and its points of local maxima, local minima, and saddles, together with the values of function $M(\mathbf{x}, t)$ are frozen into the axisymmetric fluid and gas flows.

3 Integral Invariants for Axisymmetric Flows

Remark 5: The density of the z-projection of the angular momentum of gas or fluid is $\mathcal{P}(\mathbf{x}, t) = \rho(\mathbf{x}, t)[\mathbf{x} \times \mathbf{V}(\mathbf{x}, t)] \cdot \hat{\mathbf{e}}_z$. Evidently, the function $M(\mathbf{x}, t) = [\mathbf{x} \times \mathbf{V}(\mathbf{x}, t)] \cdot \hat{\mathbf{e}}_z$ for z-axisymmetric flows is different from $\mathcal{P}(\mathbf{x}, t)$. The latter is not constant along the compressible gas streaklines.

The level sets of the z-axisymmetric function $M(\mathbf{x}, t)$ are the surfaces

$$\mathcal{S}^2_\mu(t): \qquad M(\mathbf{x}, t) = \mu = \mathrm{const}, \qquad (13)$$

which also are z-axisymmetric. The function $M(\mathbf{x}, t)$ (11) evidently vanishes if $\mathbf{x} = \lambda \hat{\mathbf{e}}_z$. Therefore, all surfaces $\mathcal{S}^2_\mu(t)$ with $\mu \neq 0$ do not intersect the axis of symmetry z. If a surface $\mathcal{S}^2_\mu(t)$ (13) does intersect the z-axis, then necessarily $\mu = 0$. If the surface $\mathcal{S}^2_\mu(t)$ is compact and smooth, then because of its axial symmetry, it is topologically either a torus T^2 or a sphere S^2.

Assume that for some μ the surface $\mathcal{S}^2_\mu(t)$ (or some part of it in case of singularities) is compact. Let $\mathcal{D}^3_\mu(t)$ be the compact three-dimensional domain bounded by the surface $\mathcal{S}^2_\mu(t)$. The conservation of the function $M(\mathbf{x}, t)$ along the fluid or gas streaklines implies that the surfaces $\mathcal{S}^2_\mu(t_1)$ of constant levels of the function $M(\mathbf{x}, t_1) = \mu$ are transported by the flow at any time $t > t_1$ into the surfaces $\mathcal{S}^2_\mu(t)$ of the same constant levels of the function $M(\mathbf{x}, t) = \mu$. Therefore, the compact domains $\mathcal{D}^3_\mu(t_1)$ are transported along the gas or fluid streaklines at any time $t > t_1$ into the compact domains $\mathcal{D}^3_\mu(t)$. In this sense, the domains $\mathcal{D}^3_\mu(t)$ and their boundary surfaces $\mathcal{S}^2_\mu(t)$ are frozen into the flow.

(c) *For z-axisymmetric, ideal, incompressible fluid flows with variable density $\rho(\mathbf{x}, t)$ and arbitrary differentiable function $G(x, y)$ of two variables x and y, there exists an infinite family of integral invariants*

$$I_{G,\mu} = \int\limits_{\mathcal{D}^3_\mu(t)} G(\rho(\mathbf{x}, t), M(\mathbf{x}, t))\mathrm{d}x, \qquad (14)$$

for any compact invariant domain $\mathcal{D}^3_\mu(t)$.

Indeed, differentiating the integral $I_{G,\mu}$ (14) with respect to time t and using the incompressibility equation $\nabla \cdot \mathbf{V} = 0$, we get

$$\frac{dI_{G,\mu}}{dt} = \int_{\mathcal{D}^3_\mu(t)} \frac{DG(\rho(\mathbf{x}, t), M(\mathbf{x}, t))}{Dt} dx.$$

Using (10), we find $dI_{G,\mu}/dt = 0$, which proves the statement (d).

For $G(x, y) = x^k y^n$, where $k, n = 1, 2, 3, \ldots$, we get the doubly infinite family of invariants $I_{k,n,\mu} = \int_{\mathcal{D}^3_\mu(t)} \rho^k(\mathbf{x}, t) M^n(\mathbf{x}, t) dx$. The invariant $I_{0,0,\mu}$ is the conserved volume of the compact domain $\mathcal{D}^3_\mu(t)$.

(d) *For any z-axisymmetric, inviscid, compressible gas flows and arbitrary differentiable functions $F(x)$, there exists an infinite family of invariants*

$$I_{F,\mu} = \int_{\mathcal{D}^3_\mu(t)} \rho(\mathbf{x}, t) F(M(\mathbf{x}, t)) dx \qquad (15)$$

for any compact invariant domain $\mathcal{D}^3_\mu(t)$.

Really, differentiating the function $I_{F,\mu}$ (15) with respect to time t and using the material derivatives and the equation $Ddx/Dt = (\nabla \cdot \mathbf{V}) dx$, see [17], we get

$$\frac{dI_{F,\mu}}{dt} = \int_{\mathcal{D}^3_\mu(t)} \left[\left(\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{V} + \rho (\nabla \cdot \mathbf{V}) \right) F(M) \right.$$

$$\left. + \rho(\mathbf{x}, t) \frac{DF(M)}{Dt} \right] dx. \qquad (16)$$

Substituting here (3) and (12), we find $dI_{F,\mu}/dt = 0$.

The integral (15) with $F(M) = 1$ defines the conserved mass of the gas (or fluid) inside the domain $\mathcal{D}^3_\mu(t)$. For $F(M) = M^n$, where $n = 1, 2, 3, \ldots$, we get a countable family of functionally independent invariants (15).

4 Applications of the Invariant $M(\mathbf{x}, t)$

The function $M(\mathbf{x}, t)$ (11) has the form

$$M(r, z, t) = \left[\mathbf{x} \times \mathbf{V}(\mathbf{x}, t) \right] \cdot \hat{\mathbf{e}}_z$$

$$= \left[\hat{\mathbf{e}}_z \times (r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z) \right] \cdot \mathbf{V}$$

$$= r\hat{\mathbf{e}}_\varphi \cdot \mathbf{V} = rw(r, z, t).$$

Hence the function $rw(r, z, t) = M(r, z, t)$ has the following physical meaning: it is the projection of the angular momentum vector of the *unit-mass* particles of gas or fluid onto the axis of symmetry z.

System (6) in cylindrical coordinates r, z, φ takes the form

$$\frac{dr}{dt} = \tilde{u}(r, z, t), \qquad \frac{dz}{dt} = \tilde{v}(r, z, t),$$

$$\frac{d\varphi}{dt} = \frac{1}{r} w(r, z, t). \qquad (17)$$

The first two equations of (17) define the induced diffeomorphisms of the poloidal plane (r, z) that preserve the function $M(r, z, t)$ (11). Let us consider the points $c_m(t) = (r_m(t), z_m(t))$, where

$$\frac{\partial M}{\partial r}(c_m(t)) = 0, \qquad \frac{\partial M}{\partial z}(c_m(t)) = 0. \qquad (18)$$

Equations (18) mean that the point $c_m(t)$ is a critical point of the function $M(r, z, t)$ restricted to the plane (r, z), see Figure 1. Since the diffeomorphisms defined by the system (17) preserve the values of the function $M(r, z, t)$, we see that points $c_m(t)$ satisfying (18) are transported by the flow diffeomorphisms. Therefore, the values of the function $M(r, z, t)$ at the points $c_m(t)$ are constant in time. Hence, we get another series of invariants

$$J_m = M(c_m(t)) = r_m(t) w(r_m(t), z_m(t), t) = \text{const}. \qquad (19)$$

We call them the functional invariants of the z-axisymmetric, inviscid fluid or gas flows. The invariants J_m (19) are defined for each point $c_m(t)$ where (18) hold.

The points $c_m(t)$ defined by (18) correspond to some z-axisymmetric circles $\mathbb{S}^1_m(t)$: $r = r_m(t)$, $z = z_m(t)$, $0 \le \varphi \le 2\pi$, where the function $M(\mathbf{x}, t)$ (11) has its local maxima, minima, or saddles. The conservation of the function $M(\mathbf{x}, t)$ along the fluid or gas streaklines implies that the extremal z-axisymmetric circles $\mathbb{S}^1_m(t)$ are transported by the flow diffeomorphisms or are frozen into the flow.

For any constant time t, consider the poloidal level curves $C_\mu(t)$ of the function $M(r, z, t)$

$$C_\mu(t): \qquad M(r, z, t) = \mu = \text{const}, \qquad \varphi = 0. \qquad (20)$$

Analogous to the surfaces $\mathcal{S}^2_\mu(t)$, the curves $C_\mu(t)$ (20) with $\mu \ne 0$ lie entirely in the domain $r > 0$, and if a curve $C_\mu(t)$ (20) does intersect the axis z ($r = 0$), then necessarily $\mu = 0$.

It is evident that the z-axisymmetric surfaces $\mathcal{S}^2_\mu(t)$ (13) are obtained by rotating the curves $C_\mu(t)$ (20) around the z-axis. Hence, if a curve $C_\mu(t)$ lies in the domain $r > 0$, is closed, and has no self-intersections, then the corresponding surface $\mathcal{S}^2_\mu(t)$ topologically is a torus $\mathbb{T}^2_\mu(t) = C_\mu(t) \times \mathbb{S}^1$,
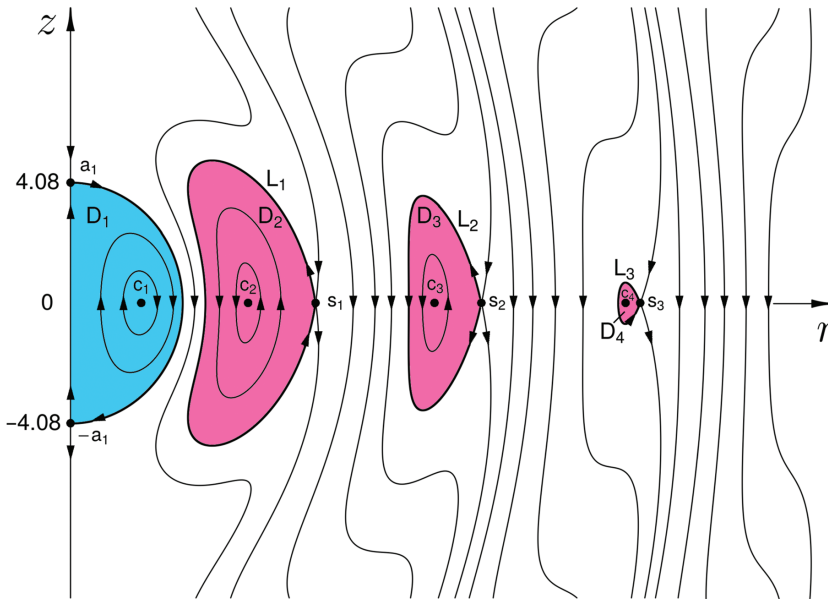
Figure 1: Poloidal section D_1 of the maximal blob $\mathcal{B}_1^3$ (blue) and poloidal sections D_2, D_3, D_4 of the maximal rings $\mathcal{R}_k^3$ (pink) for the exact z-axisymmetric fluid equilibrium with $M(r, z) = r^2 [-0.0237 + R^{-2}(R^{-1} \sin R - \cos R)]$, where $R = \sqrt{r^2 + z^2}$ (we suppose that variables r, z, R are made dimensionless by the standard scaling transform). Rotation of the domain D_1 around the z-axis defines the maximal vortex blob $\mathcal{B}_1^3$. Rotation of the domains D_2, D_3, D_4 around the z-axis defines the maximal vortex rings $\mathcal{R}_1^3$, $\mathcal{R}_2^3$, $\mathcal{R}_3^3$.

where $\mathbb{S}^1$ is the circle $0 \le \varphi \le 2\pi$. If $C_\mu(t)$ is an infinite curve in the domain $r > 0$ and has no self-intersections, then the corresponding surface $\mathcal{S}_\mu^2(t)$ (13) is an infinite cylinder $\mathbb{C}_\mu^2(t)$.

If a curve $C_0(t)$ (20) for $\mu = 0$ or some closed part of it has no self-intersections and has two intersections with the z-axis at the points $A(r = 0, z = z_1)$ and $B(r = 0, z = z_2)$, then its rotation around the z-axis defines a deformed sphere $\mathbb{S}_0^2(t)$ with two poles A and B.

If a curve $C_\mu(t)$ has no self-intersections and only one intersection with the z-axis at a point $C(r = 0, z = z_0)$, then its rotation around the z-axis defines a deformed paraboloid-like surface $\mathbb{P}_0^2(t)$ with a pole C.

Let a curve $C_\mu(t)$ (20) or some part of it (in case of self-intesections) is closed and bounds a compact set $D_\mu(t)$. It is evident that all level curves $M(r, z, t) = \text{const}$ lying inside the set $D_\mu(t)$ are also closed.

Let $C_k(t) : M(r, z, t) = \mu_k = \text{const}$ is a closed curve or a closed part of it (in case of self-intersections) that is not contained in any other compact set $D_{\mu_C}(t)$ bounded by another closed curve $M(r, z, t) = \mu_C$, see Figure 1. Let $D_k(t)$ be the compact set bounded by the closed curve $C_k(t)$. Rotation around the z-axis of the compact set $D_k(t)$ defines a three-dimensional compact set $\mathcal{D}_k^3(t) \subset \mathbb{R}^3$. If the set $\mathcal{D}_k^3(t)$ lies in the domain $r > 0$ (for example, if $\mu_k \ne 0$), then topologically it is a solid torus which we call a ring $\mathcal{R}_k^3(t) = D_k(t) \times \mathbb{S}^1$.

If a compact set $D_\ell(t)$ bounded by a curve $C_\ell(t)$: $M(r, z, t) = \mu_\ell = 0$ has a non-empty intersection I_ℓ: ($r = 0, z_1(t) \le z \le z_2(t)$) with the z-axis, then its rotation around the z-axis defines a deformed solid ball which we call (in a loose sense) a blob $\mathcal{B}_\ell^3(t)$ with two poles ($r = 0, z = z_1(t)$) and ($r = 0, z = z_2(t)$).

Remark 6: The z-axisymmetric compact rings $\mathcal{R}_k^3(t)$ and blobs $\mathcal{B}_\ell^3(t)$ are maximal in the sense that they are not contained in any larger sets for which all surfaces $\mathcal{S}_\mu^2(t)$ (13) inside them are compact.

Suppose that the fluid or gas flow in an invariant domain $\mathcal{O}$ has no singularities or discontinuities (no shock waves). Then, the fluid or gas flow in $\mathcal{O}$ defines a time-dependent family of diffeomorphisms. The conservation of the function $M(\mathbf{x}, t)$ along the fluid or gas streaklines implies that the surfaces $\mathcal{S}_\mu^2(t_1)$ of constant levels of the function $M(\mathbf{x}, t_1) = \mu$ are transported by the flow at any time $t > t_1$ into the surfaces $\mathcal{S}_\mu^2(t)$ of the same constant levels of the function $M(\mathbf{x}, t) = \mu$. Hence we get that the maximal rings $\mathcal{R}_k^3(t_1)$ and blobs $\mathcal{B}_\ell^3(t_1)$ are transformed by the flow diffeomorphisms at any time $t > t_1$ into the maximal rings $\mathcal{R}_k^3(t)$ and maximal blobs $\mathcal{B}_\ell^3(t)$. In this sense, the rings $\mathcal{R}_k^3(t)$ and blobs $\mathcal{B}_\ell^3(t)$ are frozen into the fluid or gas flows and therefore never intersect each other.

The physical meaning of the maximal rings $\mathcal{R}_k^3(t)$ and blobs $\mathcal{B}_\ell^3(t)$. The vorticity vector field corresponding

to the velocity $\mathbf{V}(r, z, t)$ (4) has the form

$$\nabla \times \mathbf{V}(r, z, t) = -\frac{1}{r}\frac{\partial(rw)}{\partial z}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(rw)}{\partial r}\hat{\mathbf{e}}_z$$
$$+ \left(\frac{\partial\tilde{u}}{\partial z} - \frac{\partial\tilde{v}}{\partial r}\right)\hat{\mathbf{e}}_\varphi. \tag{21}$$

Formula (21) implies that the vorticity lines at any constant time t lie on the surfaces of constant levels of the function $rw(r, z, t) = M(r, z, t) = \mu$. Therefore, the vorticity lines belong to the surfaces $\mathcal{S}_\mu^2(t)$ (13). All surfaces $\mathcal{S}_\mu^2(t)$ lying inside the maximal rings $\mathcal{R}_k^3(t)$ and blobs $\mathcal{B}_\ell^3(t)$ are compact, and all surfaces lying outside of them in the Euclidean space $\mathbb{R}^3$ are non-compact. Hence, the $\mathcal{R}_k^3(t)$ and $\mathcal{B}_\ell^3(t)$ have the following physical meaning for the fluid or gas flows in the Euclidean space $\mathbb{R}^3$: $\mathcal{R}_k^3(t)$ and $\mathcal{B}_\ell^3(t)$ are the maximal, compact, three-dimensional sets invariant for the vorticity lines. Therefore, we call them the maximal vortex rings $\mathcal{R}_k^3(t)$ and maximal vortex blobs $\mathcal{B}_\ell^3(t)$. Each extremal z-axisymmetric circle $\mathbb{S}_m^1(t)$ corresponding to a non-degenerate local maximum or minimum of function $M(r, z, t)$ (11) is a vortex axis frozen into the flow.

The maximal vortex rings $\mathcal{R}_k^3(t)$ and blobs $\mathcal{B}_\ell^3(t)$ lead to the integral invariants

$$R_{k.n} = \int_{\mathcal{R}_k^3(t)} \rho(\mathbf{x}, t)M^n(\mathbf{x}, t)\mathrm{d}x,$$

$$B_{\ell.n} = \int_{\mathcal{B}_\ell^3(t)} \rho(\mathbf{x}, t)M^n(\mathbf{x}, t)\mathrm{d}x, \tag{22}$$

where $n = 1, 2, 3, \ldots$, and to the functional invariants J_m (19). The proof is given in the statements (c) and (d) because the function $M(\mathbf{x}, t) = $ const on the boundaries of the maximal vortex rings and blobs.

5 Functional Independence of Integral Invariants (22) from the Helicity

Integrals (22) for any maximal domain $\mathcal{D}_k^3(t) = D_k(t) \times \mathbb{S}^1$ have the form

$$J_{k.n} = \int_{\mathcal{D}_k^3(t)} \rho(r, z, t)M^n(r, z, t)r\mathrm{d}r\mathrm{d}z\mathrm{d}\varphi$$

$$= 2\pi \int_{D_k(t)} \rho(r, z, t)r^{n+1}w^n(r, z, t)\mathrm{d}r\mathrm{d}z, \tag{23}$$

where we substituted $M(r, z, t) = rw(r, z, t)$, see Section 4.

The z-axisymmetric velocity (4) of the ideal incompressible fluid is

$$\mathbf{V}(r, z, t) = -r^{-1}\psi_z(r, z, t)\hat{\mathbf{e}}_r + r^{-1}\psi_r(r, z, t)\hat{\mathbf{e}}_z$$
$$+ w(r, z, t)\hat{\mathbf{e}}_\varphi, \tag{24}$$

where $\psi(r, z, t)$ is the stream function. The corresponding vorticity (21) becomes

$$\nabla \times \mathbf{V} = -r^{-1}(rw)_z\hat{\mathbf{e}}_r + r^{-1}(rw)_r\hat{\mathbf{e}}_z - r^{-1}$$
$$\times \left(\psi_{rr} - r^{-1}\psi_r + \psi_{zz}\right)\hat{\mathbf{e}}_\varphi.$$

Therefore, the helicity integral [8] over the domain $\mathcal{D}_k^3(t)$ is

$$H_k = \int_{\mathcal{D}_k^3(t)} \mathbf{V} \cdot (\nabla \times \mathbf{V})\mathrm{d}x$$

$$= 2\pi \int_{D_k(t)} \left[\frac{1}{r}\left[\psi_r(rw)_r + \psi_z(rw)_z\right]\right.$$

$$\left. - w\left[\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right]\right]\mathrm{d}r\mathrm{d}z. \tag{25}$$

Let us show that the integral invariants (22) are functionally independent from the helicity (25). Indeed, for any vortex ring or blob $\mathcal{D}_k^3(t)$, the integral invariants (22) and (23) as functions of the velocity $\mathbf{V}(r, z, t)$ (24) are mutually functionally independent because the integrals of the power functions $r(rw)^n$ and $r(rw)^m$ for $n \neq m$ are independent. Therefore, if for a fixed k all integrals $J_{k.n}$ (23) with arbitrary $n \geq 1$ were functionally dependent on the helicity H_k (25), then $J_{k.n}$ would have been mutually functionally dependent on each other, but they are not. This proves that the integral invariants $J_{k.n}$ (22) and (23) for all $n \geq 1$ with possibly one exception are functionally independent from the helicity H_k (25).

6 Conclusion

The following main results were obtained in this work:
- Material conservation laws (5) were derived for the axisymmetric flows of an inviscid, incompressible fluid with a variable density $\rho(\mathbf{x}, t)$. The material conservation laws (5) contain an arbitrary differentiable function $G(x, y)$ of two variables x and y and generalise the laws $F(\zeta)$ derived by Keblin et al. [5] for incompressible fluid flows with constant density ρ.
- For the axisymmetric flows of an inviscid, compressible gas, we derived the new material conservation laws (11).

– For the axisymmetric dynamics of an inviscid, compressible gas and an incompressible fluid, we proved the existence of vortex rings $\mathcal{R}_k^3(t)$ and vortex blobs $\mathcal{B}_m^3(t)$, which are frozen into the fluid and gas flows and therefore cannot intersect each other during the dynamics.

– We presented the family of functional invariants J_m, which are given by the explicit formula (19), constructed infinitely many integral invariants $R_{k.n}$ and $B_{\ell.n}$ (22), and proved that they are functionally independent from the helicity (25).

– The axisymmetric dynamics of the inviscid gas and fluid between two given states is possible only if the corresponding total numbers of vortex rings N_r are equal (the same is true for the total numbers of vortex blobs N_s) and all new invariants J_m, $R_{k.n}$, and $B_{\ell.n}$ for them coincide as well as the invariants $I_{G.\mu}$ (14) and $I_{F.\mu}$ (15).

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