

Oleg Bogoyavlenskij*

Safety Factor for the New Exact Plasma Equilibria

<https://doi.org/10.1515/zna-2018-0337>

Received July 12, 2018; accepted October 22, 2018

Abstract: An exact formula for the limit of the safety factor q at a magnetic axis is derived for the general up-down asymmetric plasma equilibria possessing axial symmetry, generalizing Bellan's formula for the up-down symmetric ones. New exact axisymmetric plasma equilibria depending on arbitrary parameters α , ξ , b_{kn} , z_{kn} , where $k = 1, \dots, M$, $n = 1, \dots, N$, are constructed ($\alpha \neq 0$ is a scaling parameter), which are up-down asymmetric in general. The equilibria are not force-free if $\xi \neq 0$ and satisfy Beltrami equation if $\xi = 0$. For some values of ξ the magnetic field and electric current fluxes have isolated invariant toroidal magnetic rings, for another ξ they have invariant spheroids (blobs) and for some values of ξ both invariant toroidal rings and spheroids (blobs). A generalization of the Chandrasekhar – Fermi – Prendergast magnetostatic model of a magnetic star is presented where plasma velocity $\mathbf{V}(\mathbf{x})$ is non-zero.

Keywords: Exact Solutions; Grad–Shafranov Equation; Helical Magnetic Field Lines; Magnetic Star Model.

1 Introduction

I. We study the equilibrium equations for a plasma with zero electrical resistance [1–3]:

$$\mathbf{J} \times \mathbf{B} = \text{grad } p, \quad \text{curl } \mathbf{B} = \mu \mathbf{J}, \quad \text{div } \mathbf{B} = 0, \quad (1)$$

where \mathbf{B} is the magnetic field, \mathbf{J} – the electric current, μ – the magnetic permeability, p – the pressure. In papers [4] and [5], Taylor asserted that at the non-zero plasma viscosity the magnetohydrodynamic turbulence leads to the plasma relaxation towards a final state where magnetic field $\mathbf{B}(\mathbf{x})$ satisfies the Beltrami equation

$$\text{curl } \mathbf{B}(\mathbf{x}) = \alpha \mathbf{B}(\mathbf{x}) \quad (2)$$

with a constant α , and has the same integral of helicity [6] as the original plasma configuration.

As known [7, 8], in the ideal magnetohydrodynamics with vanishing electrical resistance the magnetic field

$\mathbf{B}(t, \mathbf{x})$ is transformed in time by the plasma flow diffeomorphisms (or “is frozen in the flow”). Therefore, the closed magnetic field lines are transformed in time into the isotopically equivalent closed magnetic lines.¹

II. In the 1958 paper [1], Kruskal and Kulsrud proved for the plasma equilibrium (1) that a surface $p(\mathbf{x}) = \text{const}$ that “lies in a bounded domain and has no edges” in the general case is “a topological torus” that is “made up of lines of magnetic force $\mathbf{B}(\mathbf{x})$ ” and stated that “under normal circumstances each surface $p = P$ (excepting a set of values of P of measure zero) is traversed ergodically and consequently determined by any line of force contained in it.”

In the 1959 paper [9], Newcomb stated “It is easy to verify that the lines of force on a pressure surface are closed if and only if $\iota(P)/2\pi$ is rational; if it is irrational, the lines of force cover the surface ergodically.” Here $\iota(P)$ is the rotational transform connected with the safety factor $q(P)$ [10] by the relation $q(P) = 2\pi/\iota(P)$ [11, 12].

A study of safety factor q for concrete plasma equilibria may prove to be important for the problem of controlled fusion because the stability properties of a toroidal plasma equilibrium are connected with its q profile [13].

Safety factor for the time-dependent axisymmetric magnetic fields is studied in [14].

III. Axisymmetric steady magnetic fields have the form

$$\mathbf{B}(r, z) = -\frac{1}{r} \frac{\partial \psi}{\partial z} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial \psi}{\partial r} \hat{\mathbf{e}}_z + \frac{w(r, z)}{r} \hat{\mathbf{e}}_\varphi, \quad (3)$$

where $\psi(r, z)$ is a flux function, $2\pi w(r, z)$ is the magnetic flux circulation (analog of the swirl in fluid dynamics) and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\varphi$ are vectors of unit length along the axes of the cylindrical coordinates r , z , φ . The steady axisymmetric (1) were reduced in [2, 3] to the nonlinear Grad–Shafranov equation

$$\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} = r^2 \frac{dF}{d\psi} - \mathcal{G} \frac{d\mathcal{G}}{d\psi}, \quad (4)$$

where $F(\psi)$ and $\mathcal{G}(\psi)$ are arbitrary smooth functions of ψ connected with the pressure p and the circulation $2\pi w(r, z)$ by the relations

$$F(\psi) = -\mu p(r, z), \quad \mathcal{G}(\psi) = w(r, z). \quad (5)$$

¹ Two closed curves C_1 and C_2 in a domain $\mathcal{O} \subset \mathbb{R}^3$ are called isotopically equivalent if there exists a continuous in parameter τ ($0 \leq \tau \leq 1$) family of diffeomorphism \mathcal{D}_τ of the domain \mathcal{O} such that $\mathcal{D}_0(C_1) = C_1$ and $\mathcal{D}_1(C_1) = C_2$.

*Corresponding author: Oleg Bogoyavlenskij, Queen's University, Kingston, Canada, E-mail: Bogoyavl@mast.queensu.ca

In Section 2 we study the general Grad–Shafranov equation (4) and derive an exact formula for the limit of the safety factor q at a magnetic axis for all solutions to (4). The derived formula yields that the limit value q_m of the safety factor at a magnetic axis is finite and non-zero.

IV. There are three cases for that (4) becomes linear:

$$F(\psi) = \lambda\psi + C, \quad \mathcal{G}(\psi) = \alpha\psi, \quad (6)$$

$$F(\psi) = c^2\psi^2 + C, \quad \mathcal{G}(\psi) = \alpha\psi, \quad (7)$$

$$F(\psi) = \lambda\psi + C, \quad \mathcal{G}(\psi) = \alpha^2\sqrt{2\psi - \psi_0}, \quad (8)$$

where $\alpha, \lambda, C, \psi_0$ are arbitrary constants. Exact solutions for the case (7) that are global plasma equilibria modeling astrophysical jets are presented in [15, 16], where also extensive literature for the case (7) is quoted. Another plasma equilibria satisfying (7) with $C = 0$ and $\mathcal{G}(\psi) = \sqrt{\alpha^2\psi^2 + \chi^2}$ are studied in [17].

Equation (4) for the case (8) has the non-homogeneous linear form $\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \lambda r^2 - \alpha^4$. The case (8) was first found by Solov'ev [18] and later was investigated in numerous publications, see for example [19, 20].

In Sections 3–8, we study the special case (6) for which (4) takes the form

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = \alpha^2(\xi r^2 - \psi), \quad (9)$$

where we denote $\xi = \lambda\alpha^{-2}$. For the case (6), the plasma pressure p (5) takes the form (up to an additive constant)

$$\tilde{p}(r, z) = -\alpha^2\xi\mu^{-1}\psi(r, z). \quad (10)$$

Therefore, the plasma equilibria (6) are not force-free, if $\xi = \lambda\alpha^{-2} \neq 0$.

V. Equation 9 has the following two important properties:

(a) If a function $\psi(r, z)$ satisfies (9) then all functions

$$\Psi_N(r, z) = \psi(r, z) + \sum_{n=1}^N b_n \frac{\partial^n \psi(r, z)}{\partial z^n} \quad (11)$$

with arbitrary parameters $\alpha, \xi, b_1, \dots, b_N$ also satisfy (9).

Indeed, differentiating (9) n times with respect to z , we find

$$\begin{aligned} & \left[\frac{\partial^n \psi(r, z)}{\partial z^n} \right]_{rr} - \frac{1}{r} \left[\frac{\partial^n \psi(r, z)}{\partial z^n} \right]_r + \left[\frac{\partial^n \psi(r, z)}{\partial z^n} \right]_{zz} \\ &= -\alpha^2 \frac{\partial^n \psi(r, z)}{\partial z^n}. \end{aligned} \quad (12)$$

Multiplying (12) with arbitrary constants b_n , $n = 1, 2, \dots, N$, and adding with (9), we get that function $\Psi_N(r, z)$ (11) satisfies (9).

Analogously, since all functions $\partial^n \psi(r, z)/\partial z^n$ satisfy (12) we find that functions

$$\Phi_N(r, z) = \xi r^2 + \sum_{n=1}^N b_n \frac{\partial^n \psi}{\partial z^n}(r, z) \quad (13)$$

satisfy (9).

(b) If functions $\psi_k(r, z)$ satisfy (9) with $\xi = \xi_k$:

$$\psi_{krr} - \frac{1}{r}\psi_{kr} + \psi_{kzz} = \alpha^2(\xi_k r^2 - \psi_k), \quad (14)$$

then for any constants $c_1, \dots, c_M, z_1, \dots, z_M$ the functions $\Phi_M(r, z) = \sum_{k=1}^M c_k \psi_k(r, z + z_k)$ satisfy (9) with $\xi = c_1\xi_1 + \dots + c_M\xi_M$.

Indeed, (14) are invariant under translations $z \rightarrow z + z_k$. Hence functions $\psi_k(r, z + z_k)$ satisfy (14). Multiplying the latter with constants c_k and adding we get that functions $\Phi_M(r, z)$ satisfy (9) with $\xi = c_1\xi_1 + \dots + c_M\xi_M$.

As a consequence of the statements (a) and (b) we get that if a function $\psi(r, z)$ satisfies (9), then functions

$$\Psi_{N,M}(r, z) = \psi(r, z) + \sum_{k=1}^M \sum_{n=1}^N b_{kn} \frac{\partial^n \psi}{\partial z^n}(r, z + z_{kn}) \quad (15)$$

with arbitrary constant parameters b_{kn}, z_{kn} also satisfy (9).

VI. Let us study exact solution to (9) with the flux function

$$\psi = r^2 [\xi - G_2(\alpha R)], \quad (16)$$

where $R = \sqrt{r^2 + z^2}$ and

$$G_2(u) = \frac{1}{u^2} \left(\cos u - \frac{\sin u}{u} \right). \quad (17)$$

Figure 1 shows poloidal sections of magnetic surfaces $\psi(r, z) = \text{const}$ for plasma equilibrium (3), (16) with $\xi = -0.0237$ and $\alpha = 1$. They are up-down symmetric because flux function (16) satisfies the identity $\psi(r, z) = \psi(r, -z)$. This equilibrium has three **B, J**-invariant maximal magnetic rings and one **B, J**-invariant spheroid of radius $a = 4.08$. The rings are maximal in the sense that they are not contained in any bigger **B, J**-invariant rings.

We have shown in [21], that the range I^* of function $G_2(u)$ is the segment $[-1/3, \xi_1 \approx 0.02872]$. For the special values of parameter ξ belonging to the range I^* one can find such a number a that $\xi = G_2(\alpha a)$. Then the flux function (16) takes the form

$$\psi_H = r^2 [G_2(\alpha a) - G_2(\alpha R)]. \quad (18)$$

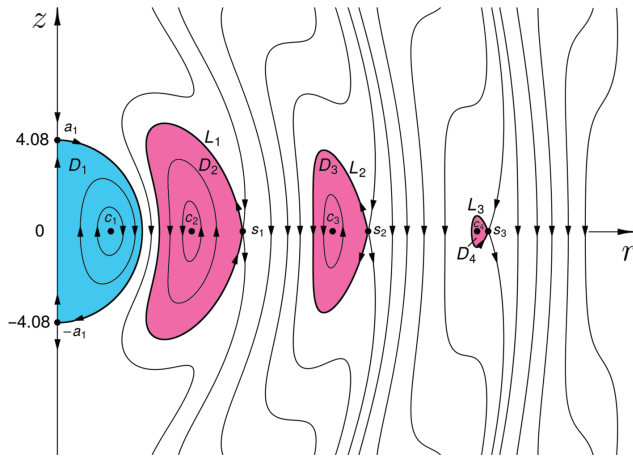


Figure 1: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = -0.0237$. Rotation of the semicircle joining points $-a_1$ and a_1 around the axis z defines \mathbf{B} , \mathbf{J} -invariant sphere $\mathbb{S}^2_{a_1}$ (blue). Rotation of the interiors of the loop separatrices L_1, L_2, L_3 around the axis z defines three \mathbf{B} , \mathbf{J} -invariant isolated magnetic rings $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ (pink) bounded by the \mathbf{B} , \mathbf{J} -invariant tori $\mathbb{T}^2 = L_i \times \mathbb{S}^1$. The rings are maximal in the sense that they are not contained in any bigger \mathbf{B} , \mathbf{J} -invariant rings.

The corresponding magnetic field \mathbf{B} (3) has invariant spheroid \mathbb{B}_a of radius a with boundary sphere defined by the equation $\psi_H(r, z) = 0$. In another form, the stream function (18) was first derived in 1899 in the pioneer paper by Hicks [22] that is the historical precursor of many works on fluid and plasma equilibria.²

Remark 1: Parameters α and ξ have the following physical meaning: parameter ξ defines the range of the safety factor $q(\psi)$ for magnetic field $\mathbf{B}(\alpha, \xi)$ in each of the $\mathbf{B}(\alpha, \xi)$ -invariant spheroids \mathbb{B}_{a_k} (for $-1/3 < \xi < \xi_1 \approx 0.02872$, $\xi = G_2(\alpha a_k)$) and characterizes the deviation of magnetic field $\mathbf{B}(\alpha, \xi)$ (3) from the spheromak magnetic field \mathbf{B}_s [24, 25] that coincides with $\mathbf{B}(\alpha, 0)$. Parameter α specifies the scaling of the plasma equilibria $\mathbf{B}(\alpha, \xi)$.

VII. Consider the special case of function $\Phi_N(r, z)$ (13):

$$\Phi_1(r, z) = \xi r^2 + \frac{1}{\alpha^2} \frac{\partial \psi(r, z)}{\partial z} = \xi r^2 - \frac{1}{\alpha^2} \frac{\partial [r^2 G_2(\alpha R)]}{\partial z}.$$

Using formula (17) we find:

$$\begin{aligned} \Phi_1(r, z) &= r^2 [\xi - z G_3(\alpha R)], \\ G_3(u) &= \frac{1}{u^4} \left[(3 - u^2) \frac{\sin u}{u} - 3 \cos u \right]. \end{aligned} \quad (19)$$

² The author is grateful to H.K. Moffatt for his suggestion to compare our results with results of Hicks paper [22], see Section 9 below. The author has found in the literature another paper by Hicks [23] that is the Abstract of [22] published separately in 1898. Therefore the two papers [22, 23] should be read together, see the footnote 4 below.

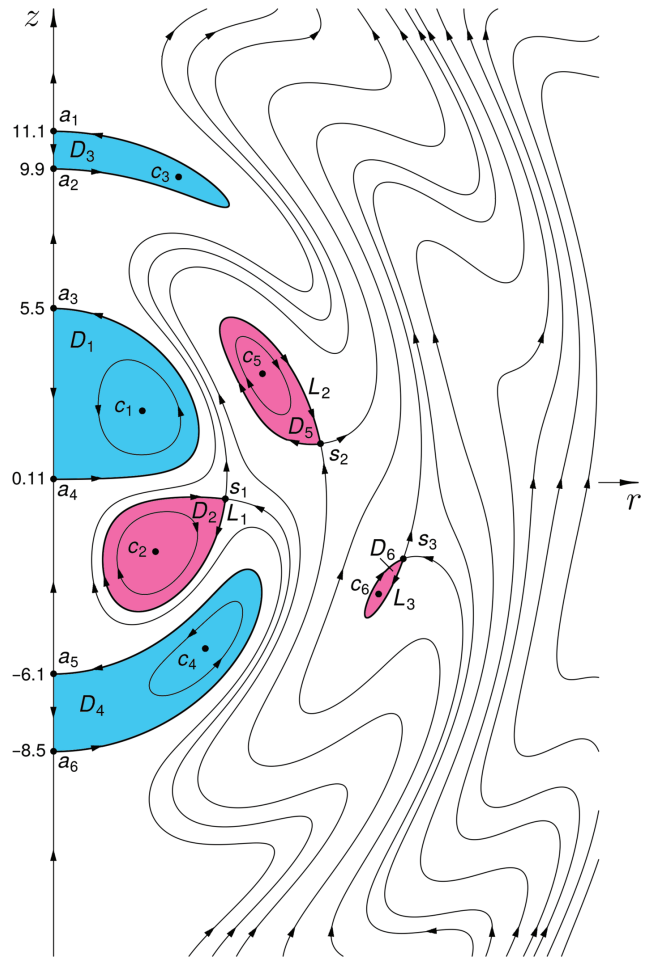


Figure 2: Poloidal contours of magnetic surfaces $\Phi_1(r, z) = \text{const}$ for the up-down asymmetric plasma equilibrium with flux function $\Phi_1(r, z) = r^2 [0.0075 - z G_3(R)]$ (19). Rotation of the contours around the axis z defines three maximal magnetic rings (pink) and three magnetic blobs (blue).

Figure 2 shows poloidal sections $\Phi_1(r, z) = \text{const}$ of magnetic surfaces for magnetic field (3), (19) for $\xi = 0.0075$ and $\alpha = 1$. Plasma equilibrium (3), (19) has three \mathbf{B} , \mathbf{J} -invariant maximal magnetic rings and three magnetic blobs. Flux function (19) and its magnetic surfaces $\Phi_1(r, z) = \text{const}$ evidently are up-down asymmetric.

The general flux function $\Psi_2(r, z)$ (11) has the form

$$\begin{aligned} \Psi_2(r, z) &= r^2 [\xi - G_2(\alpha R) + b_1 z G_3(\alpha R) \\ &\quad + b_2 (G_3(\alpha R) + z^2 \alpha^2 G_4(\alpha R))] \end{aligned} \quad (20)$$

and is up-down asymmetric if $b_1 \neq 0$. Here function $G_4(u)$ has the form

$$G_4(u) = \frac{1}{u^6} \left[(6u^2 - 15) \frac{\sin u}{u} - (u^2 - 15) \cos u \right].$$

VIII. The electric current \mathbf{J} for the magnetic field \mathbf{B} (3) with $w = \mathcal{G}(\psi)$ is

$$\begin{aligned}\mu\mathbf{J} &= \text{curl } \mathbf{B} \\ &= -\frac{[\mathcal{G}(\psi)]_z}{r}\hat{\mathbf{e}}_r + \frac{[\mathcal{G}(\psi)]_r}{r}\hat{\mathbf{e}}_z - \frac{1}{r}(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz})\hat{\mathbf{e}}_\varphi.\end{aligned}\quad (21)$$

Remark 2: Equations 3 and 21 yield that for the magnetic circulation $w(r, z) = \mathcal{G}(\psi) = \alpha\psi$ and any flux functions $\psi(r, z)$ satisfying (9) with $\xi = 0$ the magnetic field \mathbf{B} (3) satisfies the Beltrami equation $\text{curl } \mathbf{B} = \alpha\mathbf{B}$. Therefore, the flux functions $\Psi_N(r, z)$ (11) and $\Phi_N(r, z)$ (13) for $\xi = 0$ define an infinite-dimensional linear space of axisymmetric Beltrami fields (3). Lüst and Schlüter [26] were the first to study the axisymmetric Beltrami fields; exact formulas for them in terms of Bessel's functions were derived by Chandrasekhar [25], see also [27]. The flux functions (11), (13), (16) and the corresponding to them Beltrami fields (3) have explicit form in terms of elementary functions, see formulas (19), (20), (40).

Remark 3: In Section 5, we construct for infinitely many values ξ_k ($k = 1, 2, 3, \dots$) of parameter ξ exact magnetic fields $\mathbf{B}(\alpha, \xi_k)$ (3), (16) which are defined in a spheroid of certain radius a_k and together with the pressure p vanish at the boundary sphere $R = a_k$. The first eight values ξ_k are presented in (65) below. These solutions are continuously matched with empty space for $R > a_k$ and therefore have finite energy. For these solutions, the question of asymptotic behaviour at $r \rightarrow \infty$ has no interest since they vanish for all $R \geq a_k$. For other values of parameter $\xi \neq \xi_k, k = 1, 2, 3, \dots$, magnetic field $\mathbf{B}(r, z)$ (3) and electric current $\mathbf{J}(r, z)$ (21) have the following asymptotics at $r \rightarrow \infty$:

$$\mathbf{B}(r, z) \approx 2\xi\hat{\mathbf{e}}_z + \alpha\xi r\hat{\mathbf{e}}_\varphi, \quad \mathbf{J}(r, z) \approx 2\alpha\xi\mu^{-1}\hat{\mathbf{e}}_z. \quad (22)$$

Formulae (22) imply that electric current $\mathbf{J}(r, z)$ asymptotically is constant at $r \rightarrow \infty$ and magnetic field lines asymptotically are helices lying on the vertical cylinders $r = \text{const}$. For $\xi = 0$ we get the spheromak magnetic field $\mathbf{B}_s(r, z)$ [24, 25] for which both $\mathbf{B}_s(r, z)$ and $\mathbf{J}_s(r, z)$ tend to zero at $r \rightarrow \infty$.

IX. We discuss simultaneously the plasma equilibria and fluid equilibria because they obey the equivalent equations [28]. The plasma equilibria corresponding to the Hicks hydrodynamic solutions (18) were presented in 1957 by Prendergast [29] in terms of Bessel's functions. In this paper we use the definition of the pitch \mathcal{P} of helical curves (and the corresponding safety factor $q = \mathcal{P}/(2\pi)$) that is different from the Hicks' [22] and Moffatt's ones [30] and

that is applicable to any axisymmetric fluid and plasma equilibria.³

Hicks derived in [22] (4) for the steady axisymmetric stream functions $\psi(r, z)$ that is widely known as the Grad-Shafranov equation [2, 3] because they independently rediscovered it 59 years later, in 1958. The same equation was rediscovered in 1950 by the aerospace engineers Bragg and Hawthorne [36] and in 1957 by Lüst and Schlüter [37]. Hicks presented in [22] the stream function that describes the spheromak Beltrami flow (with parameter $\lambda = 0$ (6)) that in 1958 was rediscovered in terms of Bessel's functions by Woltjer [24] as a plasma equilibrium and by Chandrasekhar in [25] among many force-free axisymmetric plasma equilibria satisfying the linear Beltrami equation (2) with constant α . Hicks derived in [22] the conditions for vanishing of the corresponding to (18) flow on the boundary sphere S_a^2 rediscovered in 1957 by Prendergast [29]. Hicks' fluid equilibria defined by the ψ functions (18) were rediscovered in 1969 by Moffatt [30] in terms of Bessel's functions.⁴

³ Using his definition of the pitch, Hicks correctly described the ranges of the pitch for four special solutions (18). For example, Hicks proved that the "angular pitch" of stream lines for the spheromak Beltrami flow in the first invariant spheroid is changing in the segment $[257^\circ 27' 30'', 297^\circ 4']$ that corresponds to the range $[0.7152, 0.8252]$ for the safety factor. In 2000, P.M. Bellan derived in [13], p.85, for the spheromak Beltrami field the similar limits " $q_{\text{wall}} = 0.715$ and $q_{\text{axis}} = 0.82$ ". The steady fluid flows defined by the flux function (18) were rediscovered in 1969 by H.K. Moffatt [30] in terms of Bessel's functions. Counterexamples to Moffatt's statements on vortex [30, 31] and magnetic knots [32] are constructed in [21, 33, 34]. Moffatt's Corrigendum considering vortex knots in fluid dynamics is published in [35].

⁴ In the history of hydrodynamics there is an enigma why Hicks' paper [22], in spite of its important results, had fallen into oblivion throughout the entire XX-th century, up to now. Hicks was preoccupied in [22, 23] with applications of his exact solutions to the Kelvin's vortex theory of atoms [38]. Hicks wrote about vortex atoms in the Abstract [23] of [22], on pp.336–337:

"Nevertheless, it is possible to get general ideas. The most striking one is the fact of the periodic property of atoms."

"The metals belong to aggregates having an even number of layers, i.e. the outer rotational motion is opposite to that at the centre.

The non-metals belong to aggregates having an odd number of the same, i.e. the outer rotational motion is in the same direction as that at the centre."

"The metals of high fusibility have their stream and vortex lines nearly co-incident. The alkalis have their outer layer thin, the calcium group thicker, and so on."

The Kelvin's vortex theory of atoms was discredited at the end of the XIX-th century. This is the plausible cause why Hicks papers [22, 23] were ignored in the 1932 edition of the book [39] by his contemporary H. Lamb, who knew Hicks very well because he had referenced in [39] nine earlier papers by Hicks. G.K. Batchelor in the 1967 book [40] had

Remark 4: As known, in the ideal hydrodynamics the vortex field curl \mathbf{V} is frozen into the flow. We have shown in [21] that for the exact solutions (16), the properties of the hydrodynamic safety factor q_h defined for the vortex field curl \mathbf{V} are qualitatively different from the magneto-hydrodynamic safety factor q investigated in the present paper. In [33], we studied vortex knots for the spheromak Beltrami fluid flow that has the stream function $\psi = -r^2 G_2(\alpha R)$. We have derived exact formulas for the ranges of the rational numbers m/n that classify the vortex knots in all invariant spherical shells. In [34], the vortex knots are classified for another axisymmetric Beltrami flow that has stream function $\psi = -zr^2 G_3(\alpha R)$ that is a special case for $\xi = 0$ of the flux function $\Phi_1(r, z)$ (19). This flow has invariant plane $z = 0$ and invariant hemispheres $\mathbb{S}_{k+}^2, z \geq 0, R = R_k$ and $\mathbb{S}_{k-}^2, z \leq 0, R = R_k$ where R_k are roots of equation $G_3(\alpha R) = 0$.

X. Plasma equilibria (3), (16) with parameter ξ outside the range I^* , namely $0.02872 < \xi < 0.11182$, possess several maximal $\mathbf{B}(\alpha, \xi), \mathbf{J}(\alpha, \xi)$ -invariant magnetic rings \mathcal{R}_i^3 that are bounded by toroidal magnetic surfaces and filled with nested magnetic surfaces isomorphic to tori \mathbb{T}^2 and having the innermost magnetic axes \mathbb{S}_i^1 . The rings are maximal in the sense that they are not contained in any bigger \mathbf{B}, \mathbf{J} -invariant rings. Equilibria (3), (16) with $\xi \in [0.02872, 0.11182]$ have no \mathbf{B}, \mathbf{J} -invariant spheroids, see Figure 3 for $\xi = 0.0461$.

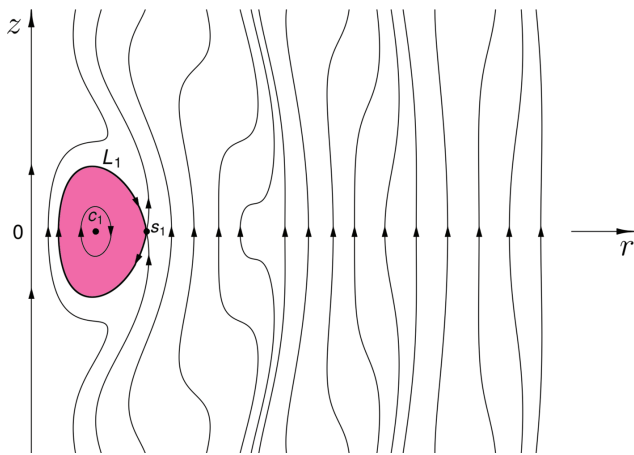


Figure 3: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = 0.0461$. Rotation of the interior of the loop separatrix L_1 around the axis z defines the maximal \mathbf{B}, \mathbf{J} -invariant magnetic ring \mathcal{R}_1 (pink) bounded by the \mathbf{B}, \mathbf{J} -invariant torus $\mathbb{T}^2 = L_1 \times \mathbb{S}^1$. All closed magnetic field lines (or magnetic knots) and all closed electric current \mathbf{J} lines are located inside the ring \mathcal{R}_1 .

mentioned none of Hicks' works. In the 2012 thesis [41] "The vortex theory of atoms" the Hicks' papers [22, 23] were not quoted in spite of his earlier papers (until 1895) were mentioned.

We show in Section 4 that all plasma equilibria (3), (16) with ξ belonging to the segment $[-0.0648, 0.02872]$, $\xi \neq 0$, inside the range I^* have several maximal \mathbf{B}, \mathbf{J} -invariant magnetic rings \mathcal{R}_i^3 and several nested \mathbf{B}, \mathbf{J} -invariant spheroids $\mathbb{B}_{a_k}^3$, see Figures 1 and 4. We present in Section 5 equilibria (3), (16) with two and five maximal $\mathbf{B}(\alpha, \xi), \mathbf{J}(\alpha, \xi)$ -invariant magnetic rings \mathcal{R}_i^3 , see Figures 5 and 6.

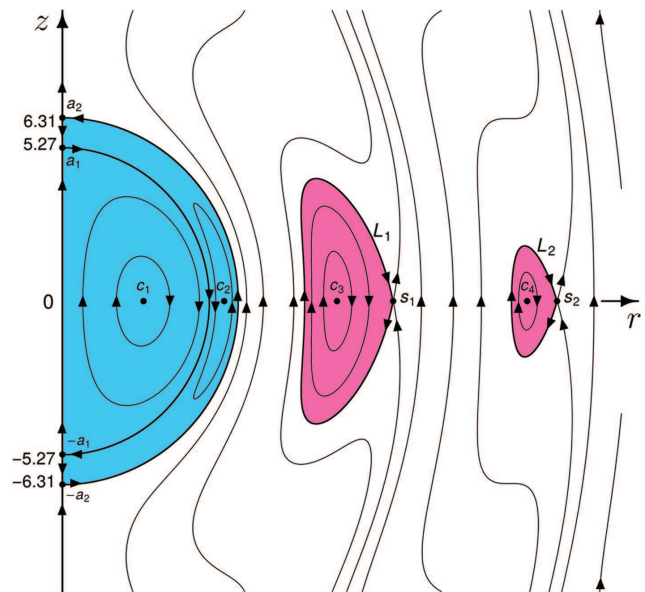


Figure 4: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = 0.025$. Rotation of the contours around the axis z defines two nested \mathbf{B}, \mathbf{J} -invariant spheroids $\mathbb{B}_{a_1}^3, \mathbb{B}_{a_2}^3$ (blue) and two maximal \mathbf{B}, \mathbf{J} -invariant magnetic rings $\mathcal{R}_1, \mathcal{R}_2$ (pink).

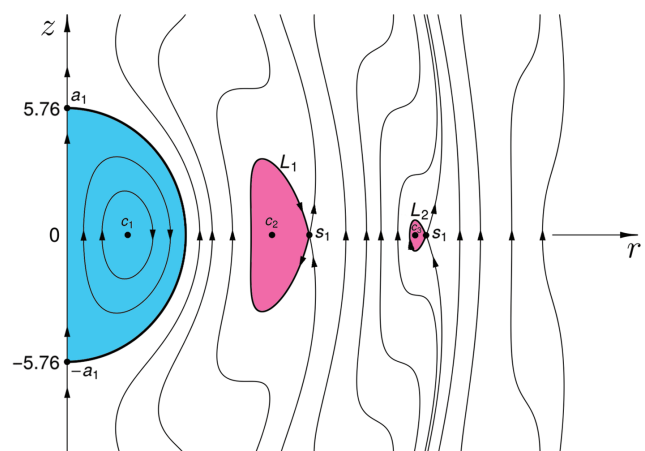


Figure 5: Poloidal contours of magnetic surfaces for $\xi = \xi_1 \approx 0.02872$. Rotation around axis z defines a \mathbf{B}, \mathbf{J} -invariant spheroid (blue) and two \mathbf{B}, \mathbf{J} -invariant maximal magnetic rings \mathcal{R}_i (pink) bounded by tori $\mathbb{T}_i^2 = L_i \times \mathbb{S}^1$. Magnetic field $\mathbf{B}(\alpha, \xi_1)$ vanishes on the sphere $\mathbb{S}_{a_1}^2$ and therefore can be continuously matched with empty space outside of it.

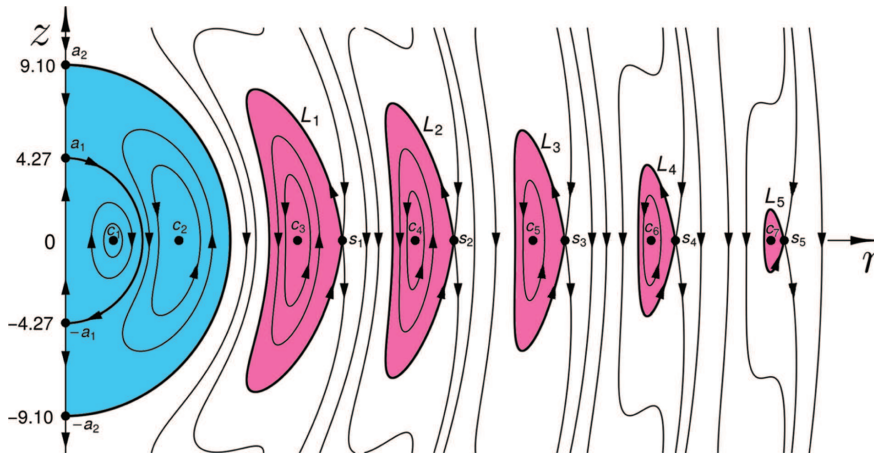


Figure 6: Poloidal contours of magnetic surfaces for $\xi = \xi_2 \approx -0.0119$. Rotation around axis z defines two nested \mathbf{B} , J -invariant spheroids (blue) and five \mathbf{B} , J -invariant maximal magnetic rings \mathcal{R}_i (pink) bounded by tori $\mathbb{T}_i^2 = L_i \times \mathbb{S}^1$. Magnetic field $\mathbf{B}(\alpha, \xi_2)$ vanishes on the second invariant sphere $\mathbb{S}_{a_2}^2$ and therefore can be continuously matched with empty space outside of it.

Remark 5: Exact flux functions (11), (13), (15), (16) define plasma equilibria which for certain values of parameters $\xi, b_{kn}, z_{kn}, k = 1, \dots, M, n = 1, \dots, N$ possess maximal magnetic rings, see Figures 1–7. These exact plasma equilibria can be used to study stability of controlled toroidal plasmas in cameras having shapes of the maximal magnetic rings. The pressure p has constant value (10) on the surface of a magnetic ring. The rings are invariant with respect to the magnetic field \mathbf{B} (3) flux and the electric current \mathbf{J} (21) flux.

Figure 8 demonstrates the poloidal contours for the force-free spheromak magnetic field (3) having the flux function (16) with $\xi = 0$ and collinear vector fields \mathbf{B} and

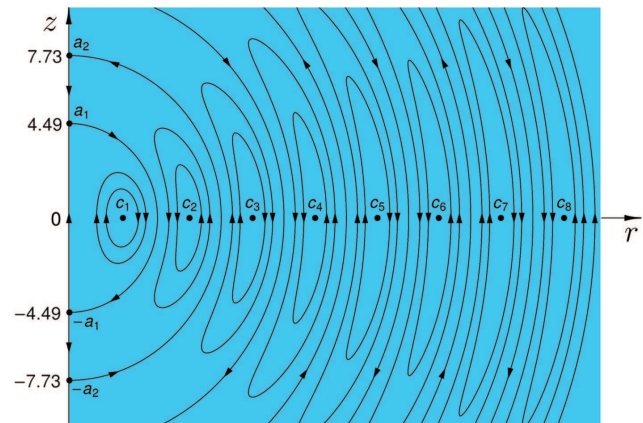


Figure 8: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = 0$ (the force-free spheromak plasma equilibrium). Rotation of the contours around the axis z defines infinitely many nested \mathbf{B} , J -invariant spheroids $\mathbb{B}_{a_k}^3$ (blue) where $R \leq a_k, k = 1, 2, 3, \dots$.

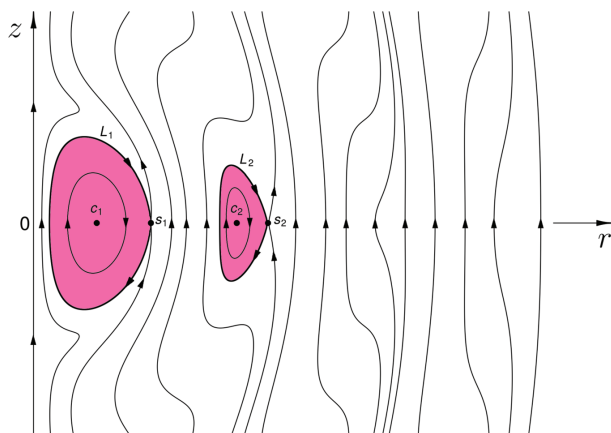


Figure 7: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = 0.0326 > \xi_1 \approx 0.02872$. Rotation of the interiors of the loop separatrices L_1, L_2 around the axis z defines two maximal \mathbf{B} , J -invariant magnetic rings $\mathcal{R}_1, \mathcal{R}_2$ (pink) bounded by the \mathbf{B} , J -invariant tori $\mathbb{T}_i^2 = L_i \times \mathbb{S}^1$. All closed magnetic field lines (or magnetic knots) and closed electric current \mathbf{J} lines are located inside the rings \mathcal{R}_1 and \mathcal{R}_2 .

\mathbf{J} satisfying Beltrami equation (2). The spheromak plasma equilibrium is qualitatively different from the non-force-free equilibria because for it the space \mathbb{R}^3 is filled with infinitely many nested $\mathbf{B}(\alpha, 0)$ -invariant spheroids and there are no isolated maximal magnetic rings.

2 Exact Limit of the Safety Factor $q(\psi)$ at a Magnetic Axis

I. Bellan derived in [13] the exact formula for the limit of the safety factor at a magnetix axis for the up-down symmetric plasma equilibria. In this section, we derive a generalization of Bellan's formula [13] for the up-down asymmetric plasma equilibria.

Remark 6: Plasma equilibria (3), (16) are up-down symmetric because flux function $\psi(r, z)$ (16) satisfies identity $\psi(r, z) = \psi(r, -z)$. The general equilibria (3), (11) and (3), (13) are up-down asymmetric if at least one coefficient b_{2k+1} is non-zero, see formulas (19), (20) and Figure 2. Plasma equilibria (3), (15) with generic constants z_{kn} also are up-down asymmetric.

As known, integral curves $\mathbf{x}(t)$ of any vector field $\mathbf{V}(\mathbf{x})$ satisfy the equation $d\mathbf{x}(t)/dt = \mathbf{V}(\mathbf{x}(t))$ [42]. Analogously the magnetic field $\mathbf{B}(\mathbf{x})$ lines (which by definition are integral curves of the vector field $\mathbf{B}(\mathbf{x})$) in the cylindrical coordinates (r, z, φ) satisfy the equation

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = \dot{r}\hat{\mathbf{e}}_r + r\dot{\varphi}\hat{\mathbf{e}}_\varphi + \dot{z}\hat{\mathbf{e}}_z = \mathbf{B}(\mathbf{x}). \quad (23)$$

Equation (23) by virtue of (3) and (5) takes the form of the system of three equations

$$\dot{r} = -\frac{1}{r}\psi_z, \quad \dot{z} = \frac{1}{r}\psi_r, \quad (24)$$

$$\dot{\varphi} = \frac{1}{r^2}\mathcal{G}(\psi). \quad (25)$$

Suppose that the flux function $\psi(r, z)$ has a local non-degenerate maximum or minimum $\psi_m = \psi(c_m)$ at a point c_m with coordinates (r_m, z_m) :

$$\begin{aligned} \psi_r(c_m) &= 0, \quad \psi_z(c_m) = 0, \\ \mathcal{H}_m &= \psi_{rr}(c_m)\psi_{zz}(c_m) - \psi_{rz}^2(c_m) > 0. \end{aligned} \quad (26)$$

Hence the system (24) has the center equilibrium point c_m and system (24)–(25) has a closed trajectory – magnetic axis $\mathbb{S}_m: r = r_m, z = z_m, 0 \leq \varphi < 2\pi$. All trajectories of system (24) near the center are closed curves $C_\psi: \psi(r, z) = \text{const}$ encircling the point c_m . The corresponding trajectories of system (24)–(25) move on the invariant axisymmetric tori $\mathbb{T}_\psi^2 = C_\psi \times \mathbb{S}^1$ (where circle \mathbb{S}^1 corresponds to the angle φ) and are either closed curves or everywhere dense on \mathbb{T}_ψ^2 infinite helices.

Assume that a closed magnetic field line on a torus \mathbb{T}_ψ^2 goes m times a long way (around the circle \mathbb{S}^1) and n times a short way (around the closed curve C_ψ). A safety factor q for such a curve is defined [10, 11] by the formula $q(\psi) = m/n$ and is the same for all magnetic lines on \mathbb{T}_ψ^2 due to the axial symmetry. For the non-closed helical magnetic lines on \mathbb{T}_ψ^2 a safety factor is $q(\psi(t_1)) = \lim m/n$ when $m, n \rightarrow \infty$ along a given infinite curve.

We will use another definition of the safety factor that is equivalent to the above. Let $t(\psi)$ be the period of the closed trajectory C_ψ . A safety factor $q(\psi)$ of the helices on

the torus \mathbb{T}_ψ^2 is equal to the increment of the angle φ during one period $t(\psi)$, divided by 2π :

$$q(\psi) = \frac{1}{2\pi} \int_0^{t(\psi)} \frac{d\varphi}{dt} dt. \quad (27)$$

Substituting (25) we get

$$q(\psi) = \frac{\mathcal{G}(\psi)}{2\pi} \int_0^{t(\psi)} \frac{dt}{r^2(t)}. \quad (28)$$

Remark 7: The safety factor $q(\psi)$ is connected with the pitch $\mathcal{P}(\psi)$ of the helical magnetic field lines by the relation $q(\psi) = \mathcal{P}(\psi)/2\pi$. The term “pitch” was used in [22, 30–32].

In the limit $\psi \rightarrow \psi_m$ we have $r(t) \rightarrow r_m$ for all t , hence (28) yields

$$q(\psi_m) = \lim_{\psi \rightarrow \psi_m} q(\psi) = t(\psi_m)\mathcal{G}(\psi_m)/(2\pi r_m^2), \quad (29)$$

where $t(\psi_m) = \lim_{\psi \rightarrow \psi_m} t(\psi)$.

II. The system of two (24) near the equilibrium point (r_m, z_m) is approximated by the system in variations [42]

$$\frac{d\delta r}{dt} = -a_{11}\delta z - a_{12}\delta r, \quad \frac{d\delta z}{dt} = a_{12}\delta z + a_{22}\delta r, \quad (30)$$

$$a_{11} = \frac{1}{r_m}\psi_{zz}(c_m), \quad a_{12} = \frac{1}{r_m}\psi_{rz}(c_m), \quad a_{22} = \frac{1}{r_m}\psi_{rr}(c_m), \quad (31)$$

where $\delta r(t) = r(t) - r_m, \delta z(t) = z(t) - z_m$. From (26), (31) we get

$$D_m = a_{11}a_{22} - a_{12}^2 = \frac{1}{r_m^2}\mathcal{H}_m > 0. \quad (32)$$

Linear system (30) has quadratic first integral

$$Q(\delta r, \delta z) = a_{22}(\delta r)^2 + 2a_{12}(\delta r)(\delta z) + a_{11}(\delta z)^2$$

that in view of (32) is either positive or negative definite. Hence its level curves $Q(\delta r, \delta z) = \text{const}$ are nested ellipses and therefore all solutions to (30) are periodic. Due to the scaling invariance of system (30) all its solutions have the same period $t_m = 2\pi/\sqrt{D_m}$.

From the general theory of differential equations [42] it follows that the limit at $\psi \rightarrow \psi_m$ of the function of periods $t(\psi)$ is the period $t_m = 2\pi/\sqrt{D_m}$ of the system in variations (30). Using formula (32) we find:

$$t(\psi_m) = \lim_{\psi \rightarrow \psi_m} t(\psi) = \frac{2\pi}{\sqrt{D_m}} = \frac{2\pi r_m}{\sqrt{\mathcal{H}_m}}. \quad (33)$$

Substituting (33) into (29) we get

$$q(\psi_m) = \lim_{\psi \rightarrow \psi_m} q(\psi) = \frac{\mathcal{G}(\psi_m)}{r_m \sqrt{\mathcal{H}_m}}. \quad (34)$$

Thus we have proved that for the case of arbitrary functions $F(\psi)$, $\mathcal{G}(\psi)$ in the Grad–Shafranov equation (4) the safety factor $q(\psi)$ has a finite and non-zero limit at $\psi \rightarrow \psi_m$ defined by formula (34) where $\mathcal{H}_m = \psi_{rr}(c_m)\psi_{zz}(c_m) - \psi_{rz}^2(c_m) > 0$ (26).

The limit (34) is one of the two exact bounds for the range of the safety factor $q(\psi)$.

Remark 8: Bellan's formula [13] for the safety factor at the magnetic axis is based on the assumption $\partial^2 \psi / \partial r \partial z(c_m) = 0$ that holds for the up-down symmetric plasma equilibria. Our formula (34) is valid for the general equilibria with $\partial^2 \psi / \partial r \partial z(c_m) \neq 0$. For example, it is valid for the new exact up-down asymmetric plasma equilibria with the flux functions $\Psi_N(r, z)$ (11) and $\Phi_N(r, z)$ (13) where at least one coefficient $b_{2k+1} \neq 0$ (see formulas (19), (20)) and for the equilibria with flux functions $\Psi_{N,M}(r, z)$ (15) with generic constants z_{kn} .

3 Magnetic Field $\mathbf{B}(\alpha, \xi)$ in the First Invariant Spheroid

I. We will use the following elementary functions

$$\begin{aligned} G_0(u) &= -\cos u, \quad G_1(u) = \frac{d}{du} G_0(u) = \frac{\sin u}{u}, \\ G_2(u) &= \frac{d}{du} G_1(u) = \frac{1}{u^2} \left(\cos u - \frac{\sin u}{u} \right), \\ G_3(u) &= \frac{d}{du} G_2(u) = \frac{1}{u^4} \left[(3 - u^2) \frac{\sin u}{u} - 3 \cos u \right], \end{aligned} \quad (35)$$

that are analytic everywhere and have the following limits at $u \rightarrow 0$:

$$G_1(0) = 1, \quad G_2(0) = -1/3, \quad G_3(0) = 1/15. \quad (36)$$

Functions $G_n(u)$ (35) are even and satisfy the identities

$$\begin{aligned} G_0(u) + G_1(u) + u^2 G_2(u) &= 0, \\ G_1(u) + 3G_2(u) + u^2 G_3(u) &= 0. \end{aligned} \quad (37)$$

The general identity

$$\begin{aligned} G_n(u) + (2n+1)G_{n+1}(u) + u^2 G_{n+2}(u) &= 0, \\ G_{n+1}(u) &= u^{-1} dG_n(u)/du \end{aligned} \quad (38)$$

follows from identities (37) by induction. Functions $G_n(u)$ (35), (38) are connected with Bessel's functions

$J_\nu(u)$ [43] of order $\nu = n - 1/2$ by the relations $G_n(u) = \sqrt{\pi/2}(-1)^{n-1} J_{n-1/2}(u)/u^{n-1/2}$.

II. Let us show that flux function $\psi(r, z) = [\xi - G_2(\alpha R)]r^2$ satisfies (9). Indeed, using formulas (38) we find

$$\begin{aligned} \psi_r &= 2r[\xi - G_2(\alpha R)] - \alpha^2 r^3 G_3(\alpha R), \\ \psi_z &= -\alpha^2 r^2 z G_3(\alpha R), \\ \psi_{rr} &= 2[\xi - G_2(\alpha R)] - 5\alpha^2 r^2 G_3(\alpha R) - \alpha^4 r^4 G_4(\alpha R), \\ \psi_{zz} &= -\alpha^2 r^2 G_3(\alpha R) - \alpha^4 r^2 z^2 G_4(\alpha R). \end{aligned}$$

Hence we get

$$\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} = -\alpha^2 r^2 [5G_3(\alpha R) + (\alpha R)^2 G_4(\alpha R)]. \quad (39)$$

Identity (38) for $n = 2$ and $u = \alpha R$ takes the form

$$G_2(\alpha R) + 5G_3(\alpha R) + (\alpha R)^2 G_4(\alpha R) = 0.$$

Substituting this identity into (39) we get

$$\psi_{rr} - \frac{1}{r} \psi_r + \psi_{zz} = \alpha^2 r^2 G_2(\alpha R) = \alpha^2 (\xi r^2 - \psi).$$

Therefore the flux function $\psi(r, z)$ (16) is an exact solution to (9).

III. Magnetic field $\mathbf{B}(\alpha, \xi)$ (3), (16) has the following explicit form

$$\begin{aligned} \mathbf{B}(\alpha, \xi) &= \alpha^2 r z G_3(\alpha R) \hat{\mathbf{e}}_r \\ &\quad + [2[\xi - G_2(\alpha R)] - \alpha^2 r^2 G_3(\alpha R)] \hat{\mathbf{e}}_z \\ &\quad + \alpha [\xi - G_2(\alpha R)] r \hat{\mathbf{e}}_\phi. \end{aligned} \quad (40)$$

Substituting $\mathcal{G}(\psi) = \alpha\psi$ and formula (16) into the system (24), (25), we get:

$$\begin{aligned} \dot{r} &= \alpha^2 r z G_3(\alpha R), \\ \dot{z} &= 2[\xi - G_2(\alpha R)] - \alpha^2 r^2 G_3(\alpha R), \end{aligned} \quad (41)$$

$$\dot{\phi} = \alpha [\xi - G_2(\alpha R)]. \quad (42)$$

Formula (40) for $\xi = 0$ describes the spheromak fluid flow \mathbf{B}_s (satisfying Beltrami equation $\text{curl } \mathbf{B} = \alpha \mathbf{B}$) that was discovered first by Hicks [22] in terms of elementary functions and half a century later was rediscovered by Woltjer [24] and Chandrasekhar [25] in terms of Bessel functions $J_{3/2}(\alpha R)$ and $J_{5/2}(\alpha R)$ as a force-free plasma equilibrium. The corresponding flux function $\psi(r, z)$ was derived in [24, 25] in the form

$$\psi(r, z) = r^2 A \left(\frac{a}{R} \right)^{3/2} J_{3/2}(\alpha R). \quad (43)$$

Hicks formula (18) with $G_2(\alpha a) = 0$ coincides with the Chandrasekhar – Woltjer one (43) for $A = \sqrt{\pi/2}(\alpha a)^{-3/2}$ in view of the formula for the Bessel function

$$\begin{aligned} J_{3/2}(u) &= -\frac{1}{\sqrt{\pi/2}\sqrt{u}} \left(\cos u - \frac{\sin u}{u} \right) \\ &= -\frac{1}{\sqrt{\pi/2}} u^{3/2} G_2(u) \end{aligned}$$

that follows from [43], p. 56. The spheromak plasma equilibrium solution was studied (in terms of the Bessel functions) in [13, 27, 32, 44–48], see review [49]. The term “spheromak” was first introduced in [44].

IV. Suppose that parameter ξ belongs to the range of function $G_2(u)$. Then there exists such a minimal number $a > 0$ that $\xi = G_2(\alpha a)$. Formula (18) yields that magnetic surface $\psi_H(r, z) = 0$ contains the sphere \mathbb{S}_a^2 . Hence the whole spheroid \mathbb{B}_a^3 where $R \leq a$ is invariant under the magnetic flux defined by the field $\mathbf{B}(\alpha, \xi)$ (40).

The first root of equation $G_2(u) = 0$ is

$$u = \bar{u}_1 \approx 4.4931. \quad (44)$$

Figure 9 shows that the range of function $y_1(u) = G_2(u)$ for all positive u is the segment $I^*: [-1/3, \xi_1]$ where ξ_1 is the maximal value of function $G_2(u)$. The maximum is attained at a point u_1 where derivative $G'_2(u_1)$ vanishes, $\xi_1 = G_2(u_1)$. From (35) we get $G_3(u_1) = u_1^{-1} G'_2(u_1) = 0$. The first root of equation $G_3(u) = 0$ is $u_1 \approx 5.7635$. Hence we find the value of $\xi_1 = G_2(u_1) \approx 0.02872$. Only for $\xi \in I^*$ equation

$$G_2(u) = \xi \quad (45)$$

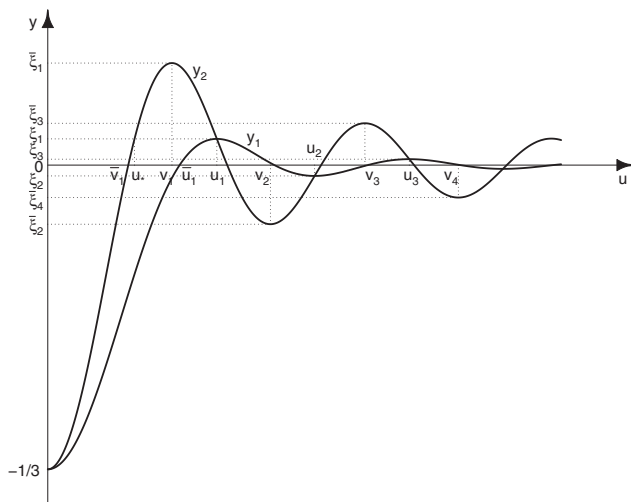


Figure 9: Plots of functions $y_1(u) = G_2(u)$ and $y_2(u) = -[G_1(u) + G_2(u)]/2$.

has some solutions. Let $g_1(\xi)$ be the smallest solution to (45):

$$G_2(g_1(\xi)) = \xi. \quad (46)$$

We will use also function

$$g(\xi) = g_1(\xi)/(2\pi). \quad (47)$$

Both functions $g_1(\xi)$ and $g(\xi)$ are defined on the segment $I^*: [-1/3, \xi_1 \approx 0.02872]$ and monotonously increase from $g_1(-1/3) = 0$ to $g_1(\xi_1) = u_1 \approx 5.7635$ and from $g(-1/3) = 0$ to $g(\xi_1) = u_1/(2\pi) \approx 0.9173$.

Function $y_1(u) = G_2(u)$ (35) near $u = 0$ has the form $G_2(u) \approx -1/3 + u^2/30$, see Figure 9. Therefore from (45) we find the asymptotics of function $g_1(\xi)$ at $\xi \rightarrow -1/3$:

$$g_1(\xi) \approx \sqrt{30}\sqrt{\xi + 1/3}, \quad g_1(-1/3) = 0. \quad (48)$$

The following values of function $g_1(\xi)$ will be used later:

$$g_1(0) = \bar{u}_1 \approx 4.4931, \quad g_1(\xi_1) = u_1 \approx 5.7635. \quad (49)$$

V. Magnetic field $\mathbf{B}(\alpha, \xi)$ (40) for $\xi \in I^*$ has an invariant sphere \mathbb{S}_a^2 of radius $a = g_1(\xi)/\alpha$ on which it takes the form

$$\tilde{\mathbf{B}}(\alpha, \xi) = \alpha^2 r G_3(g_1(\xi)) [z\hat{\mathbf{e}}_r - r\hat{\mathbf{e}}_z]. \quad (50)$$

Hence the spheroid \mathbb{B}_a^3 defined by the inequality $R \leq a = g_1(\xi)/\alpha$ is invariant under the magnetic flux (40).

Let us consider the Hill's flux function [50]

$$\psi_a(r, z) = Cr^2 \left(1 - \frac{a^3}{R^3} \right), \quad C = \text{const},$$

that satisfies equation

$$\psi_{a,rr} - \frac{1}{r} \psi_{a,r} + \psi_{a,zz} = 0. \quad (51)$$

Hence function $\psi_a(r, z)$ satisfies the Grad–Shafranov equation (4) with $F(\psi) = 0$, $\mathcal{G}(\psi) = 0$. Therefore, the corresponding magnetic field (3) has $w(r, z) = \mathcal{G}(\psi) = 0$ and takes the form

$$\mathbf{B}_a(r, z) = -3Crz \frac{a^3}{R^5} \hat{\mathbf{e}}_r + C \left[2 \left(1 - \frac{a^3}{R^3} \right) + 3r^2 \frac{a^3}{R^5} \right] \hat{\mathbf{e}}_z. \quad (52)$$

Using (21), (51) and $\mathcal{G}(\psi) = 0$ we find $\mathbf{J}_a = \text{curl } \mathbf{B}_a = 0$. Hence magnetic field \mathbf{B}_a (52) is current free.

On the sphere \mathbb{S}_a^2 ($R = a$) vector field (52) becomes

$$\tilde{\mathbf{B}}_a = \frac{-3C}{a^2} r [z\hat{\mathbf{e}}_r - r\hat{\mathbf{e}}_z]. \quad (53)$$

Vector field (53) coincides with (50) when $C = -\frac{1}{3}\alpha^2 a^2 G_3(\alpha a)$. Therefore we assume that magnetic field $\mathbf{B}(\alpha, \xi)$ (40) inside the invariant spheroid \mathbb{B}_a^3 with $a = g_1(\xi)/\alpha$ is continuously matched with the magnetic field $\mathbf{B}_a(r, z)$ (52) for $C = -\frac{1}{3}\alpha^2 a^2 G_3(\alpha a)$ outside the spheroid \mathbb{B}_a^3 . The magnetic field $\mathbf{B}_a(r, z)$ tends to a constant field $2C\hat{\mathbf{e}}_z$ when $R \rightarrow \infty$ and is current-free.

4 $\mathbf{B}(\alpha, \xi)$ -invariant Rings and Spheroids

I. From Figure 9 it is evident that equation $G_2(u) = \xi$ (45) for $\xi \neq 0$, $\xi \in I^* = (-1/3, \xi_1 \approx 0.02872)$ has a finite number $N(\xi)$ of roots and $N(\xi) \rightarrow \infty$ when $\xi \rightarrow 0$. That means the system (41)–(42) in the whole space \mathbb{R}^3 can have for $\xi \neq 0$ a finite number $N(\xi)$ of invariant spheroids $\mathbb{B}_{a_i}^3$ and it has infinitely many invariant spheroids when $\xi = 0$. We consider system (41)–(42) in the first invariant spheroid $\mathbb{B}_{a_1}^3$, $a_1 = \alpha^{-1}g_1(\xi)$, corresponding to the smallest root $g_1(\xi)$ of (45).

Figure 9 shows that (45) has no solutions if $\xi < -1/3$ or $\xi > \xi_1$. Hence the magnetic field (40) does not have any invariant spheroids \mathbb{B}_c^3 ($R \leq c$) if $\xi < -1/3$ or $\xi > \xi_1 \approx 0.02872$.

II. For $\xi \in I^*$ the system (41), (42) has at least one invariant spheroid $\mathbb{B}_{a_1}^3$, $a_1 = \alpha^{-1}g_1(\xi)$. System (41) in the invariant semi-disk D_1 ($\psi(r, z) \geq 0$, $r \geq 0$, $R \leq a_1$) has three equilibrium points: $s_1(r = 0, z = a_1)$, $s_2(r = 0, z = -a_1)$, $c_1(r = u_m/|\alpha|, z = 0)$. The equilibria s_1 and s_2 are non-degenerate saddles if $G_3(g_1(\xi)) \neq 0$. Their separatrices are: the interval $I_1 : (r = 0, -a_1 < z < a_1)$ and the arc $S_1 : (r^2 + z^2 = a_1^2, r > 0)$, see Figure 10. The equilibrium c_1 is a center, its coordinate $r = R_m = u_m(\xi)/\alpha$, where $u_m(\xi)$ is the first positive root of equation $2G_2(u) + u^2 G_3(u) = 2\xi$ (i.e. condition that $\dot{z} = 0$ in (41) at the point $(r = \alpha^{-1}u, z = 0)$). The latter in view of the second identity (37) takes the form

$$y_2(u) = -[G_1(u) + G_2(u)]/2 = \xi. \quad (54)$$

All equilibrium points of system (41) with $r \neq 0$ are $(r_k(\xi) = \alpha^{-1}u_k(\xi), z = 0)$ where $u_k(\xi)$ is a root of (54). The first root of equation $G_1(u) + G_2(u) = 0$ is

$$u_m(0) = \bar{v}_1 \approx 2.7437, \quad (55)$$

see Figure 9 for the plot of function $y_2(u) = -[G_1(u) + G_2(u)]/2$.

III. It is evident from Figure 9 that for $0 < \xi < \bar{\xi}_1$ solutions to (54) appear in pairs $u_{2k-1}(\xi) < u_{2k}(\xi)$, $k = 1$,

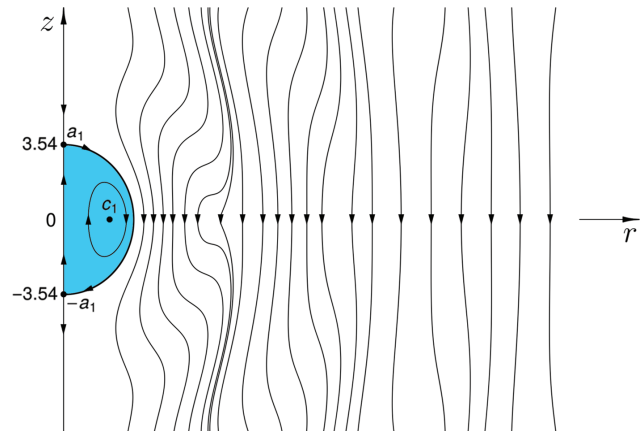


Figure 10: Poloidal contours of magnetic surfaces $\psi(r, z) = \text{const}$ (16) for $\xi = -0.0652 < \bar{\xi}_2 \approx -0.0648$. The semicircle $R = a_1$ satisfying equation $\psi(r, z) = 0$ (16) bounds domain D_1 . Rotation of the interior of the semicircle $R = a_1$ around the axis z defines \mathbf{B}, J -invariant spheroid $\mathbb{B}_{a_1}^3$ (blue).

$2, \dots, N$. From the plots of functions $y_1(u) = G_2(u)$ and $y_2(u) = -[G_1(u) + G_2(u)]/2$ in Figure 9 it follows that the amplitudes of the subsequent oscillations of function $-[G_1(u) + G_2(u)]/2$ are greater than those of $G_2(u)$. Therefore if $u_\ell(\xi)$ is the maximal solution to (45), then (54) has several pairs of solutions $u_{2k-1}(\xi) < u_{2k}(\xi)$ that are greater than $u_\ell(\xi)$. The $u_{2k-1}(\xi)$ defines a magnetic axis with coordinates $r = \alpha^{-1}u_{2k-1}(\xi)$, $z = 0$, $0 \leq \varphi \leq 2\pi$ that is a stable closed trajectory of system (41), (42). The points $u_{2k}(\xi)$ define the unstable closed trajectory $r = \alpha^{-1}u_{2k}(\xi)$, $z = 0$, $0 \leq \varphi \leq 2\pi$ of system (41), (42). The stable (with respect to the system (41), (42)) magnetic axis lies inside the $\mathbf{B}(\alpha, \xi)$ -invariant isolated magnetic ring \mathcal{R}_i^3 and the unstable one lies on its toric boundary. For all roots $u_j(\xi)$ of (54) satisfying $u_j(\xi) < u_\ell(\xi)$, equations $r = \alpha^{-1}u_j(\xi)$, $z = 0$, $0 \leq \varphi \leq 2\pi$ define stable magnetic axes \mathbb{S}_j^1 inside the largest $\mathbf{B}(\alpha, \xi)$ -invariant spheroid $\mathbb{B}_{a_\ell}^3$ where $a_\ell = \alpha^{-1}u_\ell(\xi)$.

For $-1/3 < \xi < 0$ the plot of function $y_2(u) = -[G_2(u) + G_3(u)]/2$ in Figure 9 shows that the smallest solution $u = u_{\min}(\xi)$ to (54) satisfies

$$u_{\min}(\xi) < \bar{v}_1 \approx 2.7437 < \bar{u}_1 \approx 4.4931,$$

where (44) and (55) are used and \bar{u}_1 is the first root (44) of equation $G_2(u) = 0$. Therefore the corresponding magnetic axis $r = \alpha^{-1}u_{\min}(\xi)$, $z = 0$, $0 \leq \varphi \leq 2\pi$ lies inside the first $\mathbf{B}(\alpha, \xi)$ -invariant spheroid $\mathbb{B}_{a_1}^3$ and is Tables. The other solutions to (54) appear in pairs $u_{2m}(\xi) < u_{2m+1}(\xi)$, $m = 1, 2, \dots$ and the above description is applicable. If $u_\ell(\xi)$ is the maximal solution to (45), then each pair $u_{2m}(\xi) < u_{2m+1}(\xi)$ with $u_{2m}(\xi) > u_\ell(\xi)$ defines a $\mathbf{B}(\alpha, \xi)$ -invariant isolated magnetic ring \mathcal{R}_i^3 where the $u_{2m}(\xi)$

defines a stable (*with respect to the system (41), (42)*) magnetic axis and the $u_{2m+1}(\xi)$ defines the unstable one.

The presented arguments and the plots in Figure 9 prove that all magnetic fields $\mathbf{B}(\alpha, \xi)$ (40) with $\bar{\xi}_2 < \xi < \bar{\xi}_1$ possess several isolated magnetic rings. Here $\bar{\xi}_2 \approx -0.0648$ and $\bar{\xi}_1 \approx 0.11182$, see formulas (57) below and Figure 9.

IV. From Figure 9 it follows that the range of function $y_2(u) = -[G_1(u) + G_2(u)]/2$ for all positive u is the segment $[-1/3, \bar{\xi}_1]$ where $\bar{\xi}_1$ is its maximum that is attained at a point v_1 ; hence we have $G'_1(v_1) + G'_2(v_1) = 0$. Equations (35) yield $G'_1(u) + G'_2(u) = u[G_2(u) + G_3(u)]$; therefore $G_2(v_1) + G_3(v_1) = 0$. The first root of this equation is $v_1 \approx 4.2329$. Hence, we get $\bar{\xi}_1 = -2^{-1}[G_1(v_1) + G_2(v_1)] \approx 0.11182$.

The first seven zeros of function $G_2(u) + G_3(u)$ are

$$\begin{aligned} v_1 &= 4.2329, & v_2 &= 7.5896, & v_3 &= 10.8103, \\ v_4 &= 13.9941, & v_5 &= 17.1622, & v_6 &= 20.3219, \\ v_7 &= 23.4767. \end{aligned} \quad (56)$$

These points are local maxima and minima of the function $y_2(u) = -[G_1(u) + G_2(u)]/2$, see Figure 9. The corresponding values of function $y_2(u)$ are

$$\begin{aligned} y_2(v_1) &= \bar{\xi}_1 = 0.11182, & y_2(v_2) &= \bar{\xi}_2 = -0.0648, \\ y_2(v_3) &= \bar{\xi}_3 = 0.0459, & y_2(v_4) &= \bar{\xi}_4 = -0.0355, \\ y_2(v_5) &= \bar{\xi}_5 = 0.0290, & y_2(v_6) &= \bar{\xi}_6 = -0.0245, \\ y_2(v_7) &= \bar{\xi}_7 = 0.0213. \end{aligned} \quad (57)$$

Values (57) give for the local maxima of function $y_2(u) = -(G_1(u) + G_2(u))/2$:

$$\bar{\xi}_1 > \bar{\xi}_3 > \bar{\xi}_5 > \bar{\xi}_1 \approx 0.02872; \quad \bar{\xi}_1 > \bar{\xi}_7.$$

These inequalities imply that for $\bar{\xi}_1 < \xi < \bar{\xi}_5$ there are three $\mathbf{B}(\alpha, \xi)$ -invariant isolated magnetic rings $\mathcal{R}_i^3 \subset \mathbb{R}^3$. For $\bar{\xi}_5 < \xi < \bar{\xi}_3$ there are two $\mathbf{B}(\alpha, \xi)$ -invariant rings $\mathcal{R}_1^3, \mathcal{R}_2^3$, see Figure 7 in Section 1, where $\xi = 0.0326$. For $\bar{\xi}_3 < \xi < \bar{\xi}_1$ there is one $\mathbf{B}(\alpha, \xi)$ -invariant isolated magnetic ring $\mathcal{R}_1^3 \subset \mathbb{R}^3$.

V. The extremal values of function $y_1(u) = G_2(u)$ are attained at zeros of function $G_3(u) = u^{-1}G'_2(u)$. The first eight of them are

$$\begin{aligned} u_1 &= 5.7635, & u_2 &= 9.0950, \\ u_3 &= 12.3229, & u_4 &= 15.5146, \\ u_5 &= 18.6890, & u_6 &= 21.8539, \\ u_7 &= 25.0128, & u_8 &= 28.1678. \end{aligned} \quad (58)$$

The corresponding local maxima and minima of function $y_1(u) = G_2(u)$ are (Fig. 9)

$$\begin{aligned} G_2(u_1) &= \xi_1 = 0.02872, & G_2(u_2) &= \xi_2 = -0.0119, \\ G_2(u_3) &= \xi_3 = 0.0065, & G_2(u_4) &= \xi_4 = -0.0041, \\ G_2(u_5) &= \xi_5 = 0.0029, & G_2(u_6) &= \xi_6 = -0.0021, \\ G_2(u_7) &= \xi_7 = 0.0016, & G_2(u_8) &= \xi_8 = -0.0013. \end{aligned} \quad (59)$$

Values (57) and (59) imply for the local minima of function $y_2(u) = -(G_1(u) + G_2(u))/2$:

$$\bar{\xi}_2 < \bar{\xi}_4 < \bar{\xi}_6 < \bar{\xi}_2 \approx -0.0119.$$

From Figure 9 we see that for $-1/3 < \xi < \bar{\xi}_2$ magnetic field $\mathbf{B}(\alpha, \xi)$ has only one invariant spheroid $\mathbb{B}_{a_1}^3$ and a number of isolated magnetic rings $\mathcal{R}_i^3 \subset \mathbb{R}^3$ depending on ξ . For $\xi = \bar{\xi}_2 \approx -0.0119$ there are five invariant magnetic rings \mathcal{R}_i , see Figure 6 in Section 1.

Equation (54) for $\xi > \bar{\xi}_1$ or $\xi < -1/3$ does not have any roots. Hence system (41) has no equilibrium points with $r > 0$ and no closed trajectories. Therefore magnetic field $\mathbf{B}(\alpha, \xi)$ (40) has no invariant tori \mathbb{T}^2 and no closed magnetic field lines for $\xi > \bar{\xi}_1$ or $\xi < -1/3$.

5 Applications to the Magnetic Star Model

I. Chandrasekhar and Fermi [51] and Prendergast [29] have proposed a model of magnetic star where it was supposed that plasma velocity vanishes: $\mathbf{V}(\mathbf{x}) \equiv 0$. In this section we present a generalization of the Chandrasekhar, Fermi and Prendergast model where dynamics of plasma is included. As in [51], [29], we suppose that the star has form of a spheroid \mathbb{B}_a^3 of radius a with constant mass density ρ . Additionally we assume that the plasma velocity $\mathbf{V}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ are collinear: $\mathbf{V}(\mathbf{x}) = \gamma \mathbf{B}(\mathbf{x})$ where $\gamma = \text{const}$ and that the boundary conditions

$$\mathbf{V}(\mathbf{x}) = 0, \quad \mathbf{B}(\mathbf{x}) = 0, \quad p(\mathbf{x}) = 0, \quad \text{grad } p(\mathbf{x}) = 0 \quad (60)$$

hold on the sphere \mathbb{S}_a^2 , $|\mathbf{x}| = a$. In this case, the plasma equilibrium inside the spheroid \mathbb{B}_a^3 is continuously matched with the empty space for $|\mathbf{x}| > a$, thus providing a model of magnetic star with dynamics of plasma inside it.

As known, for $\mathbf{V}(\mathbf{x}) = \gamma \mathbf{B}(\mathbf{x})$ the steady equations of ideal magnetohydrodynamics (viscosity $\nu = 0$ and plasma

diffusivity $\eta = 0$) are reduced to

$$\begin{aligned} (\gamma^2 \rho \mu - 1) \operatorname{curl} \mathbf{B} \times \mathbf{B} \\ = -\operatorname{grad} \left(\mu p + \frac{\gamma^2 \rho \mu}{2} |\mathbf{B}|^2 + \rho \mu \Phi \right), \\ \operatorname{div} \mathbf{B} = 0, \end{aligned} \quad (61)$$

where $\Phi(\mathbf{x})$ is the Newtonian gravitational potential. It is evident that (61) with pressure

$$p = (1 - \gamma^2 \rho \mu) \tilde{p} - \frac{\gamma^2 \rho}{2} |\mathbf{B}|^2 - \rho \Phi + C_1 \quad (62)$$

take the form of (1): $\mathbf{J} \times \mathbf{B} = \operatorname{grad} \tilde{p}$. Therefore, all previous results concerning the plasma equilibrium (1) are applicable to the more general magnetohydrodynamic equilibria (61) with $\gamma \neq \pm 1/\sqrt{\rho \mu}$.

II. Consider magnetic field $\mathbf{B}(\alpha, \xi)$ (40) defined by the flux function (16) where constant $\xi \in I^* = [-1/3, 0.02872]$. Here I^* is the range of function $G_2(u)$ [21]. Then there exists such parameter αa that $\xi = G_2(\alpha a)$. The magnetic surface $\psi(r, z) = [G_2(\alpha a) - G_2(\alpha R)]r^2 = 0$ (18) evidently contains the sphere \mathbb{S}_a^2 : $R = a$ and plasma pressure $\tilde{p}(r, z) = -\rho \alpha^2 \xi \mu^{-1} \psi(r, z)$ (10) vanishes on \mathbb{S}_a^2 .

For $\xi = G_2(\alpha a)$, magnetic field $\mathbf{B}(\alpha, \xi)$ (40) on the sphere $R = a$ takes the form

$$\mathbf{B}(\alpha, \xi) = \alpha^2 r G_3(\alpha a) [z \hat{e}_r - r \hat{e}_z] \quad (63)$$

((63) coincides with (50) because $G_2(g_1(\xi)) = \xi$). Therefore, the boundary conditions (60) are satisfied if

$$\xi = G_2(\alpha a), \quad G_3(\alpha a) = 0. \quad (64)$$

Since $G_3(u) = u^{-1} d G_2(u) / d u$ we get from (64) that the points αa are the points of extrema of function $G_2(u)$. From Figure 9 we see that the point αa is a local maximum of function $y_1(u) = G_2(u)$ if $G_2(\alpha a) > 0$ and it is a local minimum if $G_2(\alpha a) < 0$.

III. To find solutions to (64) we use calculation (58) of the first eight roots of equation $G_3(u) = 0$. The corresponding radii of invariant spheroids $\mathbb{B}_{a_k}^3$ are $a_k = \alpha^{-1} u_k$. Calculating the values of function $G_2(u)$ at the points u_k (58) we find the first eight values $\xi_k = G_2(u_k) = G_2(\alpha a_k)$:

$$\begin{aligned} \xi_1 \approx 0.02872, \quad \xi_2 \approx -0.0119, \quad \xi_3 \approx 0.0065, \\ \xi_4 \approx -0.0041, \quad \xi_5 \approx 0.0029, \quad \xi_6 \approx -0.0021, \\ \xi_7 \approx 0.0016, \quad \xi_8 \approx -0.0013. \end{aligned} \quad (65)$$

For $\xi = \xi_k$ (65) and the corresponding values $\alpha a_k = u_k$ (58) both (64) are satisfied. Hence we get that on the boundary sphere $\mathbb{S}_{a_k}^2$ magnetic field $\mathbf{B}(\alpha, \xi_k)(r, z)$ (63)

and plasma velocity $\mathbf{V}(r, z) = \gamma \mathbf{B}(\alpha, \xi_k)(r, z)$ are identically zero.⁵

Therefore, the magnetic field (40) and the plasma velocity can be continuously matched with the zero fields in the outer space $R \geq a_k$.

The plasma pressure $p(\mathbf{x})$ is defined by formula (62) where $\tilde{p}(r, z) = -\rho \alpha^2 \xi \mu^{-1} \psi(r, z)$ vanishes on the sphere \mathbb{S}_a^2 . Both plasma velocity $\mathbf{V}(\mathbf{x})$ and magnetic field $\mathbf{B}(\alpha, \xi_k)(r, z)$ vanish on \mathbb{S}_a^2 . The gravitational potential $\Phi(\mathbf{x}) = 4\pi|\mathbf{x}|$ is constant on the sphere \mathbb{S}_a^2 : $\Phi(\mathbf{x}) = \Phi(a_k) = 4\pi a_k$ since the density ρ is constant inside the spheroid $\mathbb{B}_{a_k}^3$. Therefore, the choice of constant $C_1 = \rho \Phi(a_k)$ in (62) yields $p(\mathbf{x}) = 0$ on the sphere $\mathbb{S}_{a_k}^2$. Hence the boundary conditions (60) are satisfied and the magnetic field $\mathbf{B}(\alpha, \xi_k)(r, z)$ (40), plasma velocity $\mathbf{V}(\alpha, \xi_k)(r, z) = \gamma \mathbf{B}(\alpha, \xi_k)(r, z)$ and the pressure

$$\begin{aligned} p = (\gamma^2 \rho \mu - 1) \frac{\rho \alpha^2 \xi}{\mu} [G_2(\alpha a_k) - G_2(\alpha R)] r^2 \\ - \frac{\gamma^2 \rho}{2} |\mathbf{B}|^2 - \rho \Phi(R) + \rho \Phi(a_k) \end{aligned} \quad (66)$$

inside the spheroid $\mathbb{B}_{a_k}^3$ are continuously matched with the empty space outside of it, thus giving a model of a magnetic star with dynamics of plasma inside it.

6 The Limit of the Safety Factor at a Magnetic Axis and Function $h(\xi)$

I. Let us derive the exact formula for the limit of the safety factor $q(\psi_m(\xi))$ (34) for solutions (16) with ξ belonging to the range J^* of function $-[G_1(u) + G_2(u)]/2$, that means $\xi \in J^* = [-1/3, \bar{\xi}_1 \approx 0.11182]$. Substituting $r^2 = R^2 - z^2$ into formula (16) and using the first identity (37) for $u = \alpha R$ we find

$$\psi(r, z) = \xi r^2 + \frac{1}{\alpha^2} [G_0(\alpha R) + G_1(\alpha R)] + z^2 G_2(\alpha R). \quad (67)$$

⁵ The two first values (65) $\xi_1 = G_2(\alpha a_1)$ and $\xi_2 = G_2(\alpha a_2)$ define solutions (40) coinciding with the Hicks solutions [22] corresponding to the special values $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ of his parameter λ [22] because they satisfy the Hicks condition of vanishing of the flux on the boundary sphere $\mathbb{S}_{a_i}^2$, $i = 1, 2$. Hicks and Prendergast who rediscovered Hicks' solutions in [29] had studied them only inside the invariant spheroids $\mathbb{B}_{a_1}^3$ and $\mathbb{B}_{a_2}^3$. Figures 5 and 6 demonstrate the poloidal contours of magnetic surfaces for these solutions in the whole Euclidean space \mathbb{R}^3 . Rotation of the bold semicircles $R = a_1$ and $R = a_2$ in Figures 5, 6 around the axis z defines the invariant spheres $\mathbb{S}_{a_i}^2$ on that the magnetic field $\mathbf{B}(\mathbf{x})$, the plasma velocity $\mathbf{V}(\mathbf{x}) = \gamma \mathbf{B}(\mathbf{x})$ and pressure $p(\mathbf{x})$ (66) vanish, $i = 1, 2$. The spheres $\mathbb{S}_{a_i}^2$ are the boundaries of the magnetic stars. Since the boundary conditions (60) hold on the spheres $\mathbb{S}_{a_i}^2$, the magnetic stars are continuously matched with the empty outer spaces $R > a_i$.

Differentiating function $\psi(r, z)$ (67) and using formulas (35) we get

$$\frac{\partial \psi}{\partial r} = r \left[2\xi + G_1(\alpha R) + G_2(\alpha R) + \alpha^2 z^2 G_3(\alpha R) \right], \quad (68)$$

$$\frac{\partial \psi}{\partial z} = z \left[G_1(\alpha R) + 3G_2(\alpha R) + \alpha^2 z^2 G_3(\alpha R) \right].$$

Function $\psi(r, z)$ (16) achieves its local maximal value $\psi_m(\xi)$ at the point $c_1(\xi) = (r_m(\xi), z_m(\xi))$. Since $\psi_r(c_1(\xi)) = 0$, $\psi_z(c_1(\xi)) = 0$, we get from (68): $z_m(\xi) = 0$ and $u_m(\xi) = \alpha r_m(\xi)$ satisfies (54) and is its smallest root. Hence $r_m(\xi)$ is the smallest radius of all existing magnetic axes. For the second derivatives of function $\psi(r, z)$ at the point $c_1(\xi)$ we find from (68)

$$\psi_{rr}(c_1(\xi)) = u_m^2(\xi) [G_2(u_m(\xi)) + G_3(u_m(\xi))],$$

$$\psi_{zz}(c_1(\xi)) = G_1(u_m(\xi)) + 3G_2(u_m(\xi)),$$

and $\psi_{rz}(c_1(\xi)) = 0$. Applying the second identity (37) we get $\psi_{zz}(c_1) = -u_m^2 G_3(u_m)$. Hence we get for the Hessian (26)

$$\mathcal{H}_m = -u_m^4(\xi) G_3(u_m(\xi)) [G_2(u_m(\xi)) + G_3(u_m(\xi))]. \quad (69)$$

Let us show that $\mathcal{H}_m > 0$. Equation (54) for $\xi = \bar{\xi}_1 \approx 0.11182$ has the root $u_m(\bar{\xi}_1) = v_1 \approx 4.2329 < g_1(0) = \bar{u}_1 \approx 4.4931$, where $G_2(\bar{u}_1) = 0$. Hence we get $G_2(u_m(\bar{\xi}_1)) = G_2(v_1) < 0$. Numerical calculation gives $G_2(v_1) \approx -0.0140$. Since $G_2(u)$ is monotonously increasing function of $u \in [-1/3, u_1 \approx 5.7635]$ we find $G_3(u_m(\xi)) = u_m^{-1}(\xi) G'_2(u_m(\xi)) > 0$. Since $-[G_2(u) + G_3(u)] = -u^{-1}[G'_1(u) + G'_2(u)]$ and function $y_2(u) = -\frac{1}{2}[G_1(u) + G_2(u)]$ is monotonously increasing for $u \in [0, v_1 \approx 4.2329]$ (see Figure 9), we find that $-[G_2(u_m(\xi)) + G_3(u_m(\xi))] > 0$. Hence we get from (69) $\mathcal{H}_m > 0$.

Substituting into (34) formulas (69), $\alpha r_m(\xi) = u_m(\xi)$ and $\mathcal{G}(\psi_m(\xi)) = \alpha \psi_m(\xi) = \alpha r_m^2(\xi) [\xi - G_2(u_m(\xi))]$, we find for the exact solutions (16)

$$\begin{aligned} h(\xi) &= \lim_{\psi \rightarrow \psi_m(\xi)} q(\psi) \\ &= \frac{\xi - G_2(u_m(\xi))}{u_m(\xi) \sqrt{-G_3(u_m(\xi)) [G_2(u_m(\xi)) + G_3(u_m(\xi))]}}, \end{aligned} \quad (70)$$

II. It is evident that expression (70) is a function of parameter ξ only, since $u_m(\xi)$ is function of ξ defined as the smallest root of (54). Function $h(\xi) = \mathcal{G}(\psi_m(\xi))$ (70) describes one of the two boundaries of the range of the fractions m/n that correspond to the torus knots $K_{m,n}$

realised by the magnetic field lines for the magnetic field (40) defined by the flux function $\psi(r, z)$ (16) for a given $\xi \in J^* = [-1/3, \bar{\xi}_1 \approx 0.11182]$.

From Figure 9 it follows that $\xi - G_2(u_m(\xi)) \geq 0$ for $\xi \in [-1/3, \bar{\xi}_1]$. Hence we get that $h(\xi) \geq 0$ for $\xi \in [-1/3, \bar{\xi}_1]$.

III. The limits (36) give $-\frac{1}{2}[G_1(0) + G_2(0)] = -1/3$. Hence we get from (54) $u_m(-1/3) = 0$, that is evident also from Figure 9. Near $u = 0$ we have from (35) $G_1(u) \approx 1 - u^2/6$ and $G_2(u) \approx -1/3 + u^2/30$. Hence (54) becomes $u_m^2(\xi) \approx 15(\xi + 1/3)$. Therefore

$$\xi - G_2(u_m(\xi)) \approx \xi + 1/3 - u_m^2(\xi)/30 \approx \frac{1}{2}(\xi + 1/3). \quad (71)$$

Since $G_3(0) = 1/15$ and $G_2(0) + G_3(0) = -4/15$ we find from (70), (71) $h(-1/3) = 0$ and get the asymptotics

$$h(\xi) \approx (\sqrt{15}/4) \sqrt{\xi + 1/3}, \quad \xi \rightarrow -1/3. \quad (72)$$

IV. The point $u_m(\bar{\xi}_1) = v_1$ is the point of maximum of function $y_2(u) = -[G_1(u) + G_2(u)]/2$, see Figure 9. Hence we get from (35) $G_2(v_1) + G_3(v_1) = v_1^{-1}[G'_1(v_1) + G'_2(v_1)] = 0$ and for $u_m(\xi) \approx v_1$ we have $G_2(u_m(\xi)) + G_3(u_m(\xi)) \approx v_1[G_3(v_1) + G_4(v_1)](u_m(\xi) - v_1)$. Therefore, using Taylor expansion we get that (54) near $\bar{\xi}_1$ takes the form $\bar{\xi}_1 - \xi \approx \frac{1}{4} v_1^2 [G_3(v_1) + G_4(v_1)](u_m(\xi) - v_1)^2$. Substituting this into (70) we find the asymptotics

$$\begin{aligned} h(\xi) &\approx \frac{\bar{\xi}_1 - G_2(v_1)}{v_1 \sqrt{2G_3(v_1)[G_3(v_1) + G_4(v_1)]^{1/4} (\bar{\xi}_1 - \xi)^{1/4}}}, \\ &\xi \rightarrow \bar{\xi}_1. \end{aligned} \quad (73)$$

Substituting $\bar{\xi}_1 \approx 0.11182$, $v_1 \approx 4.2329$, $G_2(v_1) = -G_3(v_1) \approx -0.0140$ and $G_4(v_1) \approx -0.0031$ into (73) we get the formula

$$h(\xi) \approx 0.5498(\bar{\xi}_1 - \xi)^{-1/4} \rightarrow \infty, \quad \xi \rightarrow \bar{\xi}_1. \quad (74)$$

Using (49) and substituting the values $u_m(0) = \bar{v}_1 \approx 2.7437$ and $u_m(\xi_1) = u_* \approx 2.9570$ into formula (70) we get for $\xi = 0$ and $\xi = \xi_1 \approx 0.02872$:

$$g(0) \approx 0.7152, \quad q(\psi_m(0)) = h(0) \approx 0.82524, \quad (75)$$

$$g(\xi_1) \approx 0.9173, \quad q(\psi_m(\xi_1)) = h(\xi_1) \approx 0.9305. \quad (76)$$

The plots of functions $h(\xi)$ and $g(\xi) = g_1(\xi)/(2\pi)$ (47) are shown in Figure 11. All values of the safety factor $q(\psi)$ inside the first invariant spheroid $\mathbf{B}_{a_1}^3$ belong to the interval $(g(\xi), h(\xi))$ that has the length $\ell(\xi) = h(\xi) - g(\xi)$. The maximal length $\ell_{\max} = \ell(\xi_m) \approx 0.1101104$ is achieved at $\xi_m \approx -0.0024483$ and the zero length is the limit of $\ell(\xi)$

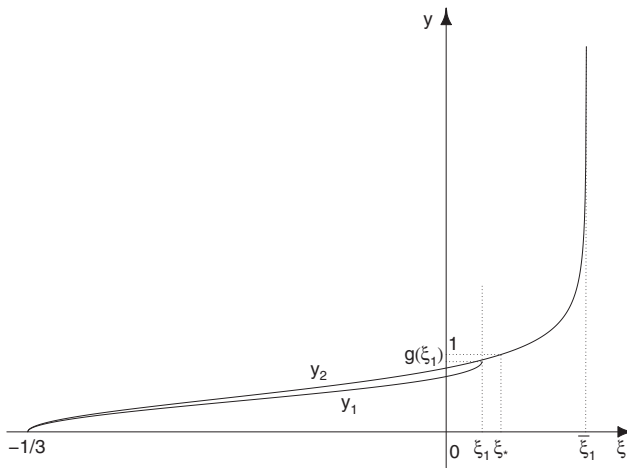


Figure 11: Plots of functions $y_1 = g(\xi)$ (47) and $y_2 = h(\xi)$ (70).

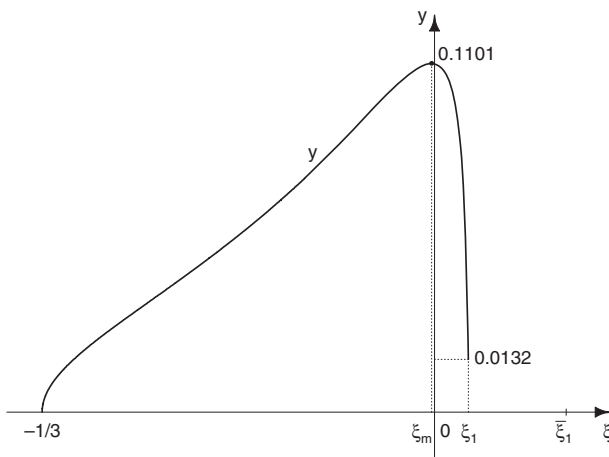


Figure 12: Plot of function $y = \ell(\xi) = h(\xi) - g(\xi)$.

at $\xi \rightarrow -1/3$. For the spheromak magnetic field $\mathbf{B}(\alpha, 0)$ we find from (75) $\ell(0) \approx 0.11002$. For $\xi = \xi_1$ we get from (76) $\ell(\xi_1) \approx 0.0132$. At the point ξ_* we have $h(\xi_*) = 1$.

The plot of function $\ell(\xi)$ is presented in Figure 12. In Figures 11 and 12 two different scales of variable y are used that have the ratio 1:35.

7 The Limit of the Safety Factor $q(\psi)$ at $\psi \rightarrow 0$

I. All trajectories of system (41) for $-1/3 < \xi \leq \xi_1$ inside domain D_1 are closed curves $C_\psi : \psi(r, z) = \psi = \text{const}$, $0 < \psi < \psi_m(\xi)$, encircling the center equilibrium point $c_1(\xi)$ and having periods $t(\psi)$, see Figure 10. The corresponding trajectories of system (41)–(42) are helices moving on invariant tori $\mathbb{T}_\psi^2 = C_\psi \times \mathbb{S}^1 \subset \mathbb{R}^3$ (circle \mathbb{S}^1 corresponds to the angle φ). In view of (42), the corresponding safety

factor $q(\psi)$ (27) is

$$q(\psi) = \frac{1}{2\pi} \int_0^{t(\psi)} \frac{d\varphi}{dt} dt = \frac{\alpha}{2\pi} \int_0^{t(\psi)} [\xi - G_2(\alpha R(t))] dt. \quad (77)$$

For $\xi \in I^*$, the closed trajectories C_ψ at $\psi \rightarrow 0$ approach the cycle of two separatrices I_1 and S_1 that satisfy equation $\psi(r, z) = 0$, see Figure 10. Since dynamics along each separatrix takes an infinite time [42] we get $\lim_{\psi \rightarrow 0} t(\psi) = \infty$. Therefore, the integral (77) contains an uncertainty because at the points $s_1(0, a_1)$, $s_2(0, -a_1)$ and on the semi-circle S^1 ($r^2 + z^2 = a_1$) we have $\xi - G_2(\alpha R(t)) = 0$, where $a_1 = g_1(\xi)/\alpha$. To resolve this uncertainty we use the invariance of the safety factor $q(\psi)$ (27) under the reparametrization of time t . After the change of time

$$\frac{d\tau}{dt} = \alpha[\xi - G_2(\alpha R)]$$

the system (41)–(42) turns into the system

$$\begin{aligned} \frac{dr}{d\tau} &= \alpha r z \frac{G_3(\alpha R)}{\xi - G_2(\alpha R)}, \\ \frac{dz}{d\tau} &= 2\alpha^{-1} - \alpha r^2 \frac{G_3(\alpha R)}{\xi - G_2(\alpha R)}, \end{aligned} \quad (78)$$

$$\frac{d\varphi}{d\tau} = 1. \quad (79)$$

On the invariant segment I_1 ($r = 0, -a_1 < z < a_1$), system (78) has the form $dr/d\tau = 0$, $dz/d\tau = 2\alpha^{-1}$. Dynamics of a trajectory along the segment I_1 from point $s_2(0, -a_1)$ to point $s_1(0, a_1)$ takes time $\tau_1 \approx 2a_1/(2\alpha^{-1}) = \alpha a_1 = g_1(\xi)$. Dynamics of a trajectory along the semicircle $R = a_1$ from point $s_1(0, a_1)$ to $s_2(0, -a_1)$ takes infinitesimally small time $\tau_2 \ll 1$ because the speed of system (78) dynamics near the circle $\xi - G_2(\alpha R) = 0$ tends to infinity.

Hence the total time $\tau_0 = \tau_1 + \tau_2$ of travel along the closed curve C_ψ has the limit $\alpha a = g_1(\xi)$ at $\psi \rightarrow 0$. During this time the total change of the angle φ is $g_1(\xi)$ because $d\varphi/d\tau = 1$, see (79). Hence we get for the limit of the safety factor (77): $\lim_{\psi \rightarrow 0} q(\psi) = g(\xi) = g_1(\xi)/(2\pi)$. The continuity of the function $q(\psi)$ implies that for $\xi \in [-1/3, \xi_1]$ the safety factor $q(\psi)$ takes all values between the two limits $q(0) = g(\xi)$ and $q(\psi_m) = h(\xi)$. Numerical calculations performed in [33] demonstrate that the safety factor $q(\psi)$ is a monotonous function of ψ , see Figure 5 of [33]. Such a monotonicity implies that the safety factor $q(\psi)$ is changing strictly inside the limits

$$g(\xi) < q(\psi) < h(\xi). \quad (80)$$

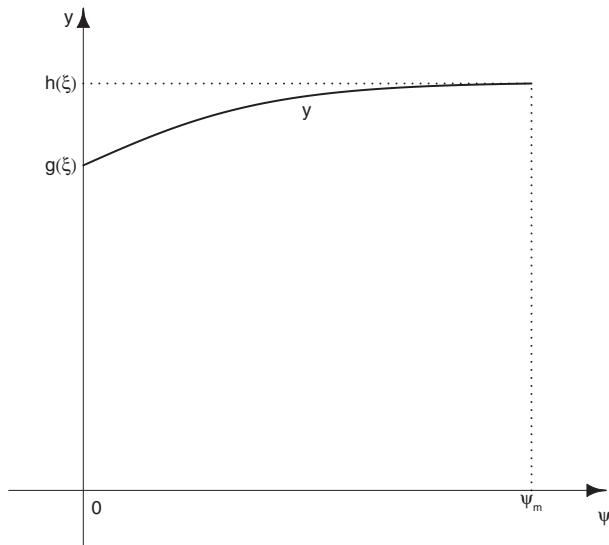


Figure 13: Plot of the safety factor $y = q(\psi)$ inside the first invariant spheroid $\mathbb{B}_{a_1}^3$ for a given ξ satisfying $-1/3 < \xi < \xi_1 \approx 0.02872$.

For example for $\xi = 0$ we get from (75):

$$0.7152 \approx g(0) < q(\psi) < h(0) \approx 0.8252. \quad (81)$$

The numerical calculations of the functions $h(\xi)$ and $g(\xi) = g_1(\xi)/(2\pi)$ are presented in their plots in Figure 11 that shows that $g(\xi) \leq h(\xi)$ for all $\xi \in I^* = [-1/3, \xi_1 \approx 0.02872]$. Hence we get that for the magnetic field knots $K_{m,n}$ realized for $\xi \in I^*$ the following inequalities hold

$$g(\xi) < \frac{m}{n} < h(\xi). \quad (82)$$

Formulas (48) and (72) imply that the functions $g(\xi)$ and $h(\xi)$ have the following asymptotics at $\xi \rightarrow -1/3$:

$$g(\xi) \approx \frac{\sqrt{30}}{2\pi} \sqrt{\xi + 1/3}, \quad h(\xi) \approx \frac{\sqrt{15}}{4} \sqrt{\xi + 1/3}. \quad (83)$$

Here $g(\xi) < h(\xi)$ because $\sqrt{30}/(2\pi) \approx 0.8717 < 0.9682 \approx \sqrt{15}/4$.

The plot of the safety factor $q(\psi)$ is shown in Figure 13. For example, for the spheromak magnetic field ($\xi = 0$) we find from (81) that only those magnetic field knots $K_{m,n}$ are realized for which the fractions m/n belong to the narrow gap of width ≈ 0.11002 defined by the inequalities $0.7152 \approx g(0) < m/n < h(0) \approx 0.8252$.

8 Safety Factor $q(\psi)$ for $\xi_1 < \xi < \bar{\xi}_1$

I. For $\xi > \bar{\xi}_1$ or $\xi < -1/3$ both (45) and (54) have no solutions, see Figure 9. Therefore, system of two (41) has no

equilibrium points and no closed trajectories. For all its trajectories at $t \rightarrow \pm\infty$ we have $z \rightarrow \pm\infty$ and $r \rightarrow \sqrt{|\psi/\xi|}$. Hence magnetic fields (3), (16) for $\xi > \bar{\xi}_1$ or $\xi < -1/3$ have no invariant tori \mathbb{T}^2 and no closed magnetic lines.

II. For $\xi_1 < \xi < \bar{\xi}_1$, (45) does not have any roots. Hence function $\psi(r, z) = r^2[\xi - G_2(\alpha R)]$ is positive for $r \neq 0$ and therefore the magnetic flux defined by formulas (3), (16) does not have any invariant spheroids \mathbb{B}_c^3 . Equation 54 for $\xi_1 < \xi < \bar{\xi}_1$ has (in general position) an even number of roots $u_{ci}(\xi) = ar_{ci}(\xi)$ and $u_{si}(\xi) = ar_{si}(\xi)$, $u_{ci}(\xi) < u_{si}(\xi)$, see Figure 9. Therefore the corresponding system of two (41) has an even number of equilibrium points: the stable centers $c_i(r = r_{ci}(\xi), z = 0)$ and the unstable saddles $s_i(r = r_{si}(\xi), z = 0)$. Each saddle equilibrium s_i has a loop separatrix L_i which has its beginning and its end at the point s_i and satisfies equation

$$\begin{aligned} \psi(r, z) &= r^2[\xi - G_2(\alpha R)] \\ &= \psi_{si}(\xi) \\ &= r_{si}^2(\xi)[\xi - G_2(u_{si}(\xi))] \\ &= \text{const} > 0. \end{aligned}$$

The phase portrait of system (41) for $\xi = 0.0326$ is shown in Figure 7, where $r_{c1} = 3.0263$, $r_{s1} = 5.6597$ and $r_{c2} = 10.1960$, $r_{s2} = 11.4485$. The loop separatrices L_1 and L_2 bound invariant domains \mathcal{D}_1 and \mathcal{D}_2 that are filled with closed trajectories C_ψ ($\psi(r, z) = \text{const}$) that define invariant tori of system of three (41), (42) $\mathbb{T}_\psi^3 = C_\psi \times \mathbb{S}^1 \subset \mathcal{D}_2 \times \mathbb{S}^1$. Here the circle \mathbb{S}^1 corresponds to the angle variable $0 \leq \varphi \leq 2\pi$. The closure of the product $\mathcal{D}_i \times \mathbb{S}^1$ is the invariant ring $\mathcal{R}_i^3 \subset \mathbb{R}^3$, $i = 1, 2$.

All magnetic field knots for the system (41), (42) are contained in the rings \mathcal{R}_i^3 because all trajectories of system (41) outside of the domains \mathcal{D}_i are infinite curves, see Figure 7.

The limit of the safety factor $q(\psi)$ at $\psi \rightarrow \psi_{mi}(\xi) = \psi(c_i(\xi))$ has the form $q(\psi_{mi}(\xi)) = h_i(\xi) > 0$ and is given by formula (70) where coordinate $u_{ci}(\xi)$ is substituted instead of $u_m(\xi)$. Closed trajectories $C_\psi \subset \mathcal{D}_i$ have period $t(\psi)$ and for $\psi \rightarrow \psi_{si}(\xi)$ approach the loop separatrix L_i . As known [42], dynamics along a loop separatrix takes an infinite time. Therefore, $\lim_{\psi \rightarrow \psi_{si}(\xi)} t(\psi) = \infty$. Equation 28 with $\mathcal{G}(\psi) = \alpha\psi$ yields for any closed trajectory C_ψ :

$$q(\psi) = \frac{\alpha\psi}{2\pi} \int_0^{t(\psi)} \frac{dt}{r^2(t)}.$$

Since $1/r^2(t) > 1/r_{si}^2(\xi)$ for all t and $\lim_{\psi \rightarrow \psi_{si}(\xi)} t(\psi) = \infty$ we get

$$q(\psi_{si}(\xi)) = \lim_{\psi \rightarrow \psi_{si}(\xi)} q(\psi) > \frac{\alpha}{2\pi r_{si}^2(\xi)} \lim_{\psi \rightarrow \psi_{si}(\xi)} (\psi t(\psi)) = \infty.$$

This result implies that the safety factor $q(\psi)$ in the domain \mathcal{D}_i is continuously changing in the limits

$$h_i(\xi) < q(\psi) < \infty.$$

Therefore for the corresponding magnetic field knots $K_{m,n}$ we have

$$h_i(\xi) < \frac{m}{n} < \infty. \quad (84)$$

Formula (74) yields the asymptotics $h_1(\xi) \approx 0.5498 (\bar{\xi}_1 - \xi)^{-1/4} \rightarrow \infty$ at $\xi \rightarrow \bar{\xi}_1$.

The equilibria $c_i(\xi)$ and $s_i(\xi)$ define respectively the stable (with respect to the system (41), (42)) and the unstable magnetic axes of the magnetic field (3), (16).

9 Comments on Hicks' Papers

I. Hicks studied in [22, 23] fluid equilibria defined by the stream function

$$\begin{aligned} \psi_H &= A \left(J_2 \left(\frac{\lambda r}{a} \right) - \frac{r^2}{a^2} J_2(\lambda) \right) \sin^2 \theta, \\ J_2(u) &= \frac{\sin u}{u} - \cos u, \end{aligned} \quad (85)$$

presented on pp. 34, 61. Here $r = \sqrt{x^2 + y^2 + z^2}$, θ is the polar angle, a the spherical aggregate's radius. On p. 34 of [22] Hicks wrote:

"The most striking and remarkable fact brought out is that with increasing parameter λ , we get a periodic system of families of aggregates. ... Of these families two are investigated more in detail than the others. In one family (the λ_2 family) all the members remain at rest in the surrounding fluid. In the other (the λ_1 family) the distinguishing feature common to all the members is that the stream lines and the vortex lines are coincident."

On p. 69 Hicks continued:

"We shall call the values of λ corresponding to the Q points the λ_2 values and denote them in order by $\lambda_2^{(1)}, \lambda_2^{(2)}, \dots, \lambda_2^{(n)}, \dots$. At $\lambda = \lambda_2^{(1)}$ the aggregate is at rest, the velocity of the fluid on the boundary is zero."

Therefore, the λ_2 family is the countable set of solutions (85) for that the fluid velocity vanishes on the boundary sphere $r = a$. The corresponding values of $\lambda = \lambda_2^{(n)}$

are roots of equation $\cot \lambda = \lambda^{-1} - \lambda/3$ [22], p. 74. These solutions were rediscovered in 1957 by Prendergast [29].⁶

Also on p. 69 Hicks wrote:

"So we get another family formed by values of λ , corresponding to points where the J -curve cuts the axis of x . We will call values of λ , corresponding to these the λ_1 parameters, and denote the orders in the same way as for the λ_2 parameters. As we shall see shortly, the distinguishing property of this family is that in each of them the vortex lines and the stream lines coincide."

Therefore, the λ_1 family is the countable set of solutions (85) where $\lambda = \lambda_1^{(n)}$ that are roots of equation $J_2(\lambda) = 0$ equivalent to $\tan \lambda = \lambda$ [22], p. 73. For these solutions the stream function (85) takes the form

$$\psi_H = A J_2 \left(\lambda_1^{(n)} r/a \right) \sin^2 \theta. \quad (86)$$

Solutions (86) were rediscovered by L. Woltjer in 1958 in terms of Bessel's functions [24] and by S. Chandrasekhar [25] in 1956 among many other axisymmetric force-free plasma equilibria. The whole family $\lambda_1^{(n)}$ actually describes the same spheromak Beltrami flow that is altered by the scaling parameter $\lambda_1^{(n)}$. As the result of the scaling by $\lambda_1^{(n)}$, solution (86) has $n - 1$ nested invariant subspheroids inside the invariant spheroid $r = a$.

The solutions (86) describe the spheromak Beltrami field that corresponds to solution (16) with $\xi = 0$.

Hence we see that Hicks had investigated not all solutions (85) but only the two subsets defined by the concrete values of $\lambda_2^{(n)}$ and $\lambda_1^{(n)}$ of which only 4 solutions defined by $\lambda_2^{(1)}, \lambda_2^{(2)}, \lambda_1^{(1)}, \lambda_1^{(2)}$ were analyzed, see pp. 93–94 of [22]. These solutions coincide with solutions (16) for three values of parameter ξ : $\xi_1 \approx 0.02872$, $\xi_2 \approx -0.0119$ and $\xi = 0$.

II. On p. 72 Hicks presented his definition of the pitch:

"The total angular pitch of the spiral is

$$\lambda \int_{x_1}^{x_2} \frac{dx}{\cos \theta}$$

where x_1, x_2 are the two roots of

$$J(\lambda x) - x^2 J(\lambda) = \frac{\lambda \psi(\text{Si } \lambda - \sin \lambda)}{\pi \mu a} = b,$$

$$\text{Si } \lambda = \int_0^\lambda \frac{\sin y}{y} dy."$$

⁶ We have demonstrated in the footnote 5 of Section 5 that the studied by Hicks special solutions (85) with parameters $\lambda = \lambda_2^{(1)}$ and $\lambda = \lambda_2^{(2)}$ correspond to the exact solutions (40) with $\xi = \xi_1 \approx 0.02872$ and $\xi = \xi_2 \approx -0.0119$ respectively.

The Hicks' definition is applicable only to the considered solutions (85). Hicks did not present any definition of the pitch for a general axisymmetric fluid equilibria.

Using his definition of pitch, after long calculations on pp. 72–92 Hicks formulates on p. 93 his main results for solutions with parameter $\lambda_2^{(1)} = 5.7637$ that corresponds to the value of our parameter $\xi = \xi_1 \approx 0.02872$:

“Angular pitch of stream lines at surface = $330^\circ 14'$.

Angular pitch of stream lines at axis = $334^\circ 58'$.

Angular pitch of vortex lines at axis = 267° .”

Dividing the Hicks' values of pitch by 360° we get that the magnetic safety factor q at the surface (of the boundary sphere \mathbb{S}_a^2) is $q = 0.9173$ and at the magnetic axis is $q = 0.9305$. These numbers coincide with $g(\xi_1)$ and $h(\xi_1)$ presented above in formulas (76). From Hicks' calculations it follows that the hydrodynamic safety factor at the vortex axis is $q_h = 267^\circ/360^\circ = 0.7417$. In Section 5 of our paper [21] we have proved that the hydrodynamic safety factor at the vortex axis for this solution is $q_h = f(\xi_1) \approx 0.7502$.

On p. 90 Hicks wrote about the pitch of vortex lines close to the boundary surface: *“The angular pitch is therefore infinite at the surface owing to the filaments being parallel to the equator at points close to the pole.”* This result is generalized in our paper [21] for any value of λ satisfying $J_2(\lambda) \neq 0$ with a clarification that limit pitch is either $+\infty$ or $-\infty$ depending on the sign of $J_2(\lambda)$.

On p. 94 Hicks presented his main results for solutions with parameter $\lambda_1^{(1)} = 4.4935$:

“Angular pitch of stream lines at surface = $257^\circ 27' 30''$.

Angular pitch of stream lines at axis = $297^\circ 4'$.”

For the safety factor at the surface we get $q = 257^\circ 27' 30''/360^\circ = 0.7152$ and at the magnetic axis $q = 297^\circ 4'/360^\circ = 0.8252$. These numbers coincide with our limits (75). Hicks presented on pp. 92–93 also formulas for pitches corresponding to parameters $\lambda_2^{(2)} = 9.0950$ and $\lambda_1^{(2)} = 7.7253$. Hicks' results are generalized for any values of λ in this paper and in [21, 33].

III. To extend the Hicks' solution (85) by the additional parameter ξ we consider the function $G_2(u)$ (17). From (17) and (85) we get $J_2(u) = -u^2 G_2(u)$. Substituting this formula and $u = \lambda R/a$ into (85) and using notations $R = \sqrt{x^2 + y^2 + z^2}$ and $R \sin \theta = r = \sqrt{x^2 + y^2}$ we get for the Hicks' solutions (85):

$$\begin{aligned}\psi_H &= -\frac{A\lambda^2}{a^2} \left(G_2\left(\frac{\lambda R}{a}\right) - G_2(\lambda) \right) R^2 \sin^2 \theta \\ &= \frac{A\lambda^2}{a^2} \left(G_2(\lambda) - G_2\left(\frac{\lambda R}{a}\right) \right) r^2.\end{aligned}$$

Inserting here $\lambda = aa$ and omitting the constant $\frac{A\lambda^2}{a^2}$ we get the formula (18): $\psi_H = [G_2(aa) - G_2(aR)]r^2$.

The constant $G_2(aa)$ evidently belongs to the range I^* of function $G_2(u)$ and hence takes values only from the segment $I^* : [-1/3, 0.02872]$ [21]. We have studied in this paper the more general fluid flows and magnetic fields with flux functions $\psi = [\xi - G_2(aR)]r^2$ (16), where ξ is the additional parameter taking all real values, $\xi \in (-\infty, \infty)$.

10 Concluding Remarks

I. We have shown in Section 2 that for any axisymmetric steady magnetic field $\mathbf{B}(r, z)$ the limit of the safety factor $q(\psi)$ at a stable (with respect to the system (41), (42)) magnetic axis has finite value $q(\psi_m)$ (34).

If safety factor $q(\psi)$ is a rational number m/n then any magnetic field line on the invariant torus $\mathbb{T}_\psi^2 = C_\psi \times \mathbb{S}^1$ makes n complete turns around the meridian C_ψ and m complete turns around the longitude \mathbb{S}^1 of the torus. Hence all helices on the “rational” tori \mathbb{T}_ψ^2 are closed curves. They traditionally are called in the literature on plasma physics “torus knots $K_{m,n}$ ”, see for example [32]. Thus the rational values of the safety factor $q(\psi)$ describe different torus knots realized for the system (24)–(25). Therefore, the classification of the (axisymmetric) magnetic field $\mathbf{B}(r, z)$ knots is reduced to the study of the range of the safety factor $q(\psi)$.

II. As known since papers by Kruskal and Kulsrud [1] and Newcomb [9], for any plasma equilibria $\mathbf{B}(\mathbf{x})$, $p(\mathbf{x})$ all closed magnetic field lines are torus knots $K_{m,n}$ which are non-trivial unless $m/n = N$ or $m/n = 1/N$ where N is an integer. All other torus knots $K_{m,n}$ and $K_{\bar{m},\bar{n}}$ with $m/n \neq \bar{m}/\bar{n}$ and $m/n \neq \bar{n}/\bar{m}$ are isotopically non-equivalent, see footnote 1 and [52]. Therefore, formula (82) implies that the set of non-equivalent magnetic knots $K_{m,n}$ belonging to the first invariant spheroid $\mathbb{B}_{a_1}^3$ for magnetic fields $\mathbf{B}(\alpha, \xi)$ (3), (16) with $\xi \in [-1/3, \xi_1 \approx 0.02872]$ coincides with the set of all rational numbers m/n satisfying the relations

$$g(\xi) < \frac{m}{n} < h(\xi), \quad \frac{m}{n} \neq \frac{1}{N}. \quad (87)$$

The functions $g(\xi) \geq 0$ and $h(\xi) \geq 0$ are given by the exact formulas (46)–(47) and (70) and satisfy the conditions $g(-1/3) = h(-1/3) = 0$, $g(\xi_1) \approx 0.9173$, $h(\xi_1) \approx 0.9305$. The plots of functions $g(\xi)$ and $h(\xi)$ in Figure 11 show that the gap $\ell(\xi) = h(\xi) - g(\xi)$ between them is very narrow. Figure 12 represents the plot of function $\ell(\xi)$ and implies that the length $\ell(\xi) < 0.11012$ for all values of ξ .

III. For $\xi = 0$ the plasma equilibrium $\mathbf{B}(0)$ (3), (16) is the spheromak Beltrami field \mathbf{B}_s [22, 24, 25]. From (81) we get that the pitch function $\mathcal{P}(\psi) = 2\pi q(\psi)$ is changing in the limits

$$4.4931 \approx 2\pi g(0) < p(\psi) < 2\pi h(0) \approx 5.1849 \quad (88)$$

that were first obtained by Hicks [22] in the angular units as $257^\circ 27' 30'' < p(\psi) < 297^\circ 4'$. Hicks did not discuss the structure of knots and was preoccupied with applications of his solutions to the Kelvin's theory of vortex atoms [38], see footnote 4 in Section 1.

Equation 88 yields that the set of isotopically non-equivalent magnetic knots $K_{m,n}$ inside the first invariant spheroid $\mathbb{B}_{a_1}^3$ ($a_1 \approx 4.4931/\alpha$) coincides with the set of all rational numbers m/n satisfying the inequalities

$$0.7152 \approx g(0) < \frac{m}{n} < h(0) \approx 0.8252. \quad (89)$$

Therefore far from all torus knots $K_{m,n}$ are realized as magnetic knots for the spheromak field \mathbf{B}_s but only those for which the safety factor $q = m/n$ belongs to the interval of a small length $\ell(0) \approx 0.11002$ defined by the inequalities (89).

IV. Hicks in [22, 23], Prendergast in [29] and Moffatt in [30–32] had studied solutions (18) only inside the invariant spheroids \mathbb{B}_a^3 . We have investigated more general solutions (16) in the whole Euclidean space \mathbb{R}^3 and have discovered the maximal \mathbf{B} , \mathbf{J} -invariant magnetic rings $\mathcal{R}_i = L_i \times \mathbb{S}^1$.

We have shown in Section 4 that all solutions (16) with parameter ξ in the range $\bar{\xi}_2 < \xi < \bar{\xi}_1$, $\xi \neq 0$, possess several magnetic rings. Here $\bar{\xi}_2 \approx -0.0648$ and $\bar{\xi}_1 \approx 0.11182$. The poloidal contours $\psi(r, z) = \text{const}$ of magnetic rings for the up-down symmetric plasma equilibria $\mathbf{B}(\alpha, \xi)$ (40) with concrete values of ξ are demonstrated in Figures 1, 3, 4, 5, 6, and 7 and for the up-down asymmetric plasma equilibria (3), (19) in Figure 2.

V. The following main results were obtained in this article:

- New exact not-force-free axisymmetric plasma equilibria depending on arbitrary parameters α , ξ , b_{kn} , z_{kn} , where $k = 1, \dots, M$, $n = 1, \dots, N$, are constructed. The corresponding flux functions $\Psi_N(r, z)$ (11), $\Phi_N(r, z)$ (13), $\Psi_{N,M}(r, z)$ (15) and $\psi(r, z)$ (16) are given in terms of elementary functions. For $\xi = 0$ they describe an infinite-dimensional space of axisymmetric Beltrami fields (previously studied in terms of Bessel's functions in [26], [25], [27]).
- We derived in Section 2 the formula (34) for the magnetic safety factor at a stable (*with respect to the system* (41), (42)) magnetic axis for the general up-down asymmetric plasma equilibria satisfying the Grad–Shafranov (4)

with arbitrary functions $F(\psi)$, $G(\psi)$. Formula (34) generalizes Bellan's formula [13] that is applicable to the up-down symmetric equilibria.

- We demonstrated in Sections 6 and 7 that for any $\xi \in I^* = [-1/3, \xi_1 \approx 0.02872]$ the complete range of the safety factor q in the first invariant spheroid $\mathbb{B}_{a_1}^3$ for all Hicks' solutions (18) is the interval $I(\xi) = (g(\xi), h(\xi))$ (80) where functions $g(\xi)$, $h(\xi)$ are presented in the explicit form. We proved that the lengths of the intervals $I(\xi)$ are bounded $|I(\xi)| \leq 0.11011$ uniformly for all $\xi \in I^*$ and the maximum 0.11011 is attained at $\xi \approx -0.0024484$. Hicks had studied in [22] only special cases corresponding to $\xi = 0.02872$, $\xi = -0.0119$ and $\xi = 0$.
- We studied the magnetic fields \mathbf{B} (3), (16) in the whole Euclidean space \mathbb{R}^3 and discovered the \mathbf{B} , \mathbf{J} -invariant isolated maximal magnetic rings, see Figures 1, 2, 3, 4, 5, 6, and 7. The rings are maximal in the sense that they are not contained in any bigger \mathbf{B} , \mathbf{J} -invariant rings. Hicks, Prendergast and Moffatt had studied in [22, 29–32] some special solutions only inside the invariant spheroids \mathbb{B}_a^3 .
- We extended the Hicks' solutions (18) to the larger family (16) depending on two parameters α , ξ that take all real values from $(-\infty, \infty)$ and studied the corresponding magnetic fields. We classified in Sections 7 and 8 the magnetic knots depending on the values of α , ξ by presenting the ranges (82) and (84) of the rational parameter m/n that distinguishes the torus knots.
- In Section 5 we extended the Chandrasekhar and Fermi [51] and Prendergast [29] magnetostatic ($\mathbf{V}(\mathbf{x}) = 0$) model of magnetic star by assuming that plasma velocity $\mathbf{V}(\mathbf{x})$ does not vanish but is proportional to the magnetic field $\mathbf{B}(\mathbf{x})$: $\mathbf{V}(\mathbf{x}) = \gamma \mathbf{B}(\mathbf{x})$, $\gamma \neq \pm 1/\sqrt{\rho\mu}$. The solutions satisfy the vanishing boundary conditions $\mathbf{V}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) = 0$, $p(\mathbf{x}) = 0$ on the surface of the invariant spheroid \mathbb{B}_a^3 and are continuously matched with the empty outer space $R > a$.

Acknowledgements: The author thanks H.K. Moffatt for his suggestion to compare our results with results of Hicks' paper [22]. The author thanks the referees for useful remarks.

References

- [1] M. D. Kruskal and R. M. Kulsrud, *Phys. Fluids*, **1**, 265 (1958).
- [2] H. Grad and H. Rubin, *Hydromagnetic equilibria and force-free fields*, in: *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy*, 31, United Nations, Geneva 1958, p. 190.
- [3] V. D. Shafranov, *Sov. Phys. JETP* **6**, 545 (1958).

- [4] J. B. Taylor, *Phys. Rev. Letters*. **33**, 1139 (1974).
- [5] J. B. Taylor, *Rev. Modern Phys.* **3**, 741 (1986).
- [6] L. Woltjer, *Proc. Nat. Acad. Sci.* **44**, 489 (1958).
- [7] W. A. Newcomb, *Ann. Phys.* **3**, 347 (1958).
- [8] T. G. Cowling, *Magnetohydrodynamics*, Adam Hilger Limited, London 1976.
- [9] W. A. Newcomb, *Phys. Fluids*. **2**, 362 (1959).
- [10] C. Mercier, *Fusion Nucleaire* **4**, 213 (1964).
- [11] J. P. Freidberg, *Plasma Physics and Fusion Energy*, Cambridge University Press 2007.
- [12] P. M. Bellan, *Fundamentals of Plasma Physics*, Cambridge University Press 2008.
- [13] P. M. Bellan, *Spheromaks. A Practical Application of Magnetohydrodynamic Dynamos and Plasma Self-Organization*, Imperial College Press 2000.
- [14] O. Bogoyavlenskij, *Z. Naturforsch.* **73**, 539 (2018).
- [15] O. I. Bogoyavlenskij, *Phys. Rev. Lett.* **84**, 1914 (2000).
- [16] O. I. Bogoyavlenskij, *J. Math. Phys.* **41**, 2043 (2000).
- [17] L. Guazzotto and J. F. Freidberg, *Phys. Plasmas* **14**, 112508 (2007).
- [18] L. S. Solov'ev, *Sov. Phys. JETP* **26**, 400 (1968).
- [19] S. B. Zheng, A. J. Wootton, and E. R. Solano, *Phys. Plasmas* **3**, 1176 (1996).
- [20] A. J. Cerfon and J. F. Freidberg, *Phys. Plasmas* **17**, 032502 (2010).
- [21] O. Bogoyavlenskij, *Phys. Rev. E* **95**, 043104 (2017).
- [22] W. M. Hicks, *Phil. Trans. Roy. Soc. London A* **192**, 33 (1899).
- [23] W. M. Hicks, *Proc. Royal Soc. London* **62**, 332 (1898).
- [24] L. Woltjer, *Bull. Astron. Inst. Netherlands* **14**, 39 (1958).
- [25] S. Chandrasekhar, *Proc. Nat. Acad. Sci.* **42**, 1 (1956).
- [26] R. Lüst and A. Schlüter, *Zeitschrift für Astrophysik*, **34**, 263 (1954).
- [27] J. Cantarella, D. DeTurck, H. Gluck, and M. Teytel, *Phys. Plasmas* **7**, 2766 (2000).
- [28] S. Chandrasekhar and K. H. Prendergast, *Proc. Natl. Acad. Sci. U.S.A.* **42**, 5 (1956).
- [29] K. H. Prendergast, *I. Astrophys. J.* **123**, 498 (1957).
- [30] H. K. Moffatt, *J. Fluid Mech.* **35**, 117 (1969).
- [31] H. K. Moffatt, Generalised vortex rings with and without swirl, In: *Vortex motion. Proceedings of the IUTAM Symposium on Fundamental Aspects of Vortex Motion*, Tokyo, Japan, 31.08–4.09, 1987, North-Holland, Amsterdam 1988, p. 22.
- [32] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, Cambridge 1978.
- [33] O. Bogoyavlenskij, *J. Math. Analys. Appl.* **450**, 21 (2017).
- [34] O. Bogoyavlenskij, *J. Math. Phys.* **58**, 1 (2017).
- [35] H. K. Moffatt, *J. Fluid Mech.* **830**, 821 (2017).
- [36] S. L. Bragg and W. R. Hawthorne, *J. Aeronautical Sciences*, **17**, 243 (1950).
- [37] R. Lüst and A. Schlüter, *Z. Naturforsch.* **12a**, 850 (1957).
- [38] W. Thomson, (Lord Kelvin) *Proc. Roy. Soc. Edin.* **6**, 94 (1867).
- [39] H. Lamb, *Hydrodynamics*, 6th ed., Cambridge University Press, Cambridge 1932.
- [40] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge 1967.
- [41] S. Van der Laan, *The Vortex Theory of Atoms, Thesis in History and Philosophy of Science*, Utrecht University 2012.
- [42] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York 1956.
- [43] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge 1980.
- [44] M. N. Rosenbluth and M. N. Bussac, *Nuclear Fusion* **19**, 489 (1979).
- [45] Z. Yoshida and Y. Giga, *Mathematische Zeitschrift*. **204**, 235 (1990).
- [46] G. F. Torres del Castillo, *J. Math. Phys.* **35**, 499 (1994).
- [47] G. E. Marsh, *Force-Free Magnetic Fields. Solutions, Topology and Applications*, World Scientific Publishing Co., Singapore 1996.
- [48] E. C. Morse, *J. Math. Phys.* **46**, 1 (2005).
- [49] T. R. Jarboe, *Plasma. Phys. Control. Fusion* **36**, 945 (1994).
- [50] M. J. M. Hill, *Phil. Trans. Roy. Soc. A* **185**, 213 (1894).
- [51] S. Chandrasekhar and F. Fermi, *Astroph. J.* **118**, 116 (1953).
- [52] A. Kawauchi, *A Survey of Knot Theory*, Birkhauser-Verlag, Basel, Boston, Berlin 1996.