DE GRUYTER Z. Naturforsch. 2019; 74(3)a: 245–251

Oleg Bogoyavlenskij*

Invariants and Conserved Quantities for the Helically Symmetric Flows of an Inviscid Gas and Fluid with Variable Density

https://doi.org/10.1515/zna-2018-0504
Received November 11, 2018; accepted December 12, 2018; previously published online January 11, 2019

Abstract: New material conservation laws and conserved quantities are derived for the helically symmetric flows of an inviscid compressible gas and an ideal incompressible fluid with variable density $\rho(\mathbf{x},\ t)$.

Keywords: Functional Invariants; Invariant Functions $\Psi_F(\mu)$; Material Conservation Laws.

1 Introduction

It is known for a long time (see [1]) that the two-dimensional ideal incompressible fluid mechanics possesses an infinite series of conserved quantities, see also [2, 3]. These are based on the material conservation laws which by definition are functions that are constant along the incompressible fluid streaklines. For the axisymmetric flows of ideal incompressible fluid with constant density ρ the first material conservation law was discovered by Hicks [4], p. 97, and rediscovered by Batchelor [2], p. 544. Later, it was rediscovered again by Kelbin et al. [5], p. 351, who also presented the material conservation laws containing an arbitrary function $F(\zeta)$ of one variable for axisymmetric and helically symmetric flows of ideal incompressible fluid with constant density ρ.

Investigations of the helically symmetric plasma equilibria were initiated by Johnson et al. [6]. Exact helically symmetric plasma equilibria were found by Bogoyavlenskij [7], together with their applications to astrophysical jets.

Helically symmetric flows were applied in [8–10] to the study of tornado-like dynamics of atmospheric gas.

The helically symmetric gas and fluid flows are invariant under the helical transformations $H_\tau : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which are the compositions of translations along the axis z and simultaneous rotations around the axis z:

$$H_\tau : \qquad z \to z + \gamma\tau, \qquad \varphi \to \varphi + \tau, \qquad r \to r, \quad (1)$$

where γ is a non-zero constant and the real parameter τ defines the Lie group of transformations (1). Here $(r,\ z,\ \varphi)$ are the cylindrical coordinates $r = \sqrt{x^2 + y^2}$, z, $\varphi = \arctan(y/x)$.

The helically symmetric flows are invariant also under the discrete subgroup of translations

$$T_N : \qquad z \to z + 2\pi\gamma N, \qquad \varphi \to \varphi, \qquad r \to r, \quad (2)$$

which are obtained from (1) for $\tau = 2\pi N$, $N = 0, \pm 1, \pm 2, \cdots$. Therefore, all helically symmetric flows are periodic in variable z with period $2\pi\gamma$.

Variable $u = z - \gamma\varphi$ is invariant under the transformations (1). All components of the helically symmetric flows depend on three variables r, $u = z - \gamma\varphi$, t and are represented by some differentiable functions $F(r, u, t)$. The 2π-periodicity of the angular variable φ implies that the functions $F(r, u, t)$ must be periodic in variable $u = z - \gamma\varphi$ with period $2\pi\gamma$, for $\gamma \neq 0$.

The helically symmetric velocity vector fields $\mathbf{V}(\mathbf{x}, t)$ have the form

$$\mathbf{V}(r, u, t) = \tilde{u}(r, u, t)\hat{\mathbf{e}}_r + \tilde{v}(r, u, t)\hat{\mathbf{e}}_z + w(r, u, t)\hat{\mathbf{e}}_\varphi, \quad (3)$$

where $u = z - \gamma\varphi$ and $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_z, \hat{\mathbf{e}}_\varphi$ are vectors of unit length in directions of coordinates r, z, φ.

Remark 1: It is evident that any helically invariant set $S \subset \mathbb{R}^3$ is non-compact. Indeed, the invariance under the helical transformations (1) implies invariance under the infinite subgroup of translations (2). Therefore, any helically invariant set S contains infinite sequences of points $T_N(x)$ going to $\pm\infty$ in variable z as $N \to \pm\infty$. Hence, any helically invariant three-dimensional domain D^3 has infinite volume and integral of any helically invariant function $F(r, u, t)$ over the domain D^3 is equal to $\pm\infty$ or zero.

In Section 2, we derive material conservation laws $F(M_\gamma(\mathbf{x}, t))$ containing an arbitrary differentiable function $F(x)$ for the helically symmetric flows of the inviscid compressible gas obeying a baroclinic equation of state.

***Corresponding author: Oleg Bogoyavlenskij,** Department of Mathematics, Queen's University, Kingston, Canada, E-mail: bogoyavl@mast.queensu.ca

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

For the helically symmetric flows of an ideal incompressible fluid with variable density $\rho(\mathbf{x}, t)$, we present in Section 2 the material conservation laws $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ containing arbitrary differentiable functions $G(x, y)$ of two variables x and y. This result is a generalisation for the case of variable density $\rho(\mathbf{x}, t)$ of the results by Kelbin et al. [5] for the helically symmetric flows with constant density ρ. The material conservation laws presented in [5] depend on an arbitrary function $F(\zeta)$ of one variable ζ. Our method is different from the methods employed in [2, 4] and [5, 11] and uses only the helical invariance of the pressure $p(\mathbf{x}, t)$.

In paper [12] we defined conserved quantities for the axisymmetric gas and fluid flows as integrals over compact three-dimensional axisymmetric domains (frozen into the flow) of the products of the gas or fluid density with material conservation laws. The construction [12] is not applicable to the helically symmetric flows because any helically invariant set $S \subset \mathbb{R}^3$ (three-, two- or one-dimensional) is non-compact and, therefore, an integral of any helically symmetric function over any helically invariant set S is either $\pm\infty$ or zero, see Remark 1.

In Section 3 we define new conserved quantities $\Psi_F(\mu)$, $\Psi_G(\mu)$ for the helically symmetric flows of inviscid compressible gas and an ideal incompressible fluid as integrals over special compact two-dimensional sets D^2_μ of the products of density $\rho(\mathbf{x}, t)$ with material conservation laws $F(M_\gamma(\mathbf{x}, t))$, $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$. The sets D^2_μ belong to the plane $z = 0$ and hence are not frozen into the gas or fluid flows and are not invariant under the helical transformations (1). Therefore, the new construction of conserved quantities $\Psi_F(\mu)$, $\Psi_G(\mu)$ is completely different from those defined in [12] for the axisymmetric case. The new invariants $\Psi_F(\mu)$, $\Psi_G(\mu)$ are piece-wise differentiable functions of a real variable μ.

Serre proved that "*any conservation law* $\int_M \phi(\mathbf{u}(x, t),$ grad $\mathbf{u}(x, t))\mathrm{d}x =$ constant *of the three-dimensional Euler equations of the incompressible perfect fluid is a linear combination of momenta, energy and helicity. Therefore, there are no other invariants of the first order (i.e. involving* $\mathbf{u}$ *and its first spatial derivatives) than those which are already known*", [13], p. 105. We introduce in this paper new invariants "*of the first order*" which do not belong to the class of conservation laws studied by Serre because they explicitly depend on the spatial variables x_1, x_2, x_3 and, therefore, are functionally independent of the momenta, kinetic energy, and helicity.

As known [14], helicity is connected with vortex knots and vortex links. The vortex knots for the axisymmetric fluid equilibria were investigated by Moffatt [14] and by Bogoyavlenskij [15–17], see also Corrigendum [18]. We introduced the problem of classification of vortex knots for the barotropic flows of inviscid compressible gas. This is an extension of Kelvin's problem [19, 20] on the vortex knots classification for the ideal incompressible fluid flows with constant density ρ.

We study Euler's equations of the inviscid adiabatic compressible gas dynamics [2]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p, \tag{4}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0, \tag{5}$$

$$\frac{\partial s}{\partial t} + \mathbf{V}\cdot\nabla s = 0, \tag{6}$$

obeying a baroclinic equation of state $p = p(s, \rho)$. Here and below $\mathbf{V}(\mathbf{x}, t)$ is the gas or fluid velocity, $p(\mathbf{x}, t) =$ the pressure, $\rho(\mathbf{x}, t) =$ the mass density, $s(\mathbf{x}, t) =$ the entropy density and $\nabla =$ the nabla operator.

We study also helically symmetric solutions of Euler's equations of the ideal incompressible fluid mechanics.

$$\begin{gathered}\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p, \quad \nabla\cdot\mathbf{V} = 0,\\ \frac{\partial \rho}{\partial t} + \mathbf{V}\cdot\nabla\rho = 0\end{gathered} \tag{7}$$

with variable density $\rho(\mathbf{x}, t)$.

2 Material Conservation Laws for Helically Symmetric Flows

(a) Let $\mathbf{V}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ *and a helically invariant pressure* $p(\mathbf{x}, t) = p(r, u, t)$ *describe an inviscid compressible gas flow. Then for any differentiable functions* $F(x)$ *the composed functions*

$$\begin{gathered}F(M_\gamma(\mathbf{x}, t)),\\ M_\gamma(\mathbf{x}, t) = \left[\mathbf{x}\times\mathbf{V}(\mathbf{x}, t) + \gamma\mathbf{V}(\mathbf{x}, t)\right]\cdot\hat{\mathbf{e}}_z\end{gathered} \tag{8}$$

are material conservation laws that means they are constant along the compressible gas streaklines.

Indeed, the gas and fluid streaklines satisfy the system of equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{V}(\mathbf{x}, t). \tag{9}$$

Let us prove first that the material derivative $\mathrm{D}M_\gamma(\mathbf{x}, t)/\mathrm{D}t$ vanishes. Our proof is equally applicable

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

for the compressible gas flows and incompressible fluid flows because we do not use here the continuity equation (5). Differentiating function $M_\gamma(\mathbf{x}, t)$ with respect to system (9) we derive

$$\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t}$$
$$= \left[\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times \mathbf{V}(\mathbf{x}, t) + \mathbf{x} \times \frac{\mathrm{D}\mathbf{V}(\mathbf{x}, t)}{\mathrm{D}t} + \gamma\frac{\mathrm{D}\mathbf{V}(\mathbf{x}, t)}{\mathrm{D}t}\right] \cdot \hat{\mathbf{e}}_z$$
$$= \left[\mathbf{V} \times \mathbf{V} + \mathbf{x} \times \left(\frac{\partial \mathbf{V}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathbf{V}}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)\right.$$
$$\left. + \gamma\left(\frac{\partial \mathbf{V}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathbf{V}}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)\right] \cdot \hat{\mathbf{e}}_z$$
$$= \left[\mathbf{x} \times \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right) + \gamma\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right)\right] \cdot \hat{\mathbf{e}}_z.$$

Substituting here Euler's equation (4) we find

$$\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t} = \left[-\frac{1}{\rho}(\mathbf{x} \times \nabla p) + \gamma\left(-\frac{1}{\rho}\nabla p\right)\right] \cdot \hat{\mathbf{e}}_z. \quad (10)$$

Applying the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$ to (10) we transform it into

$$\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t} = -\frac{1}{\rho}\left[(\hat{\mathbf{e}}_z \times \mathbf{x}) + \gamma\hat{\mathbf{e}}_z\right] \cdot \nabla p$$
$$= -\frac{1}{\rho}\left[\mathbf{U}(\mathbf{x}) + \gamma\hat{\mathbf{e}}_z\right] \cdot \nabla p, \quad (11)$$

where $\mathbf{U}(\mathbf{x}) = \hat{\mathbf{e}}_z \times \mathbf{x}$. The vector field $\mathbf{U}(\mathbf{x})$ generates the one-parametric group of rotations around the axis z with the unit angular velocity. Therefore $\mathbf{U}(\mathbf{x}) \cdot \nabla p = \nabla_{\mathbf{U}(\mathbf{x})} p = \partial p/\partial\varphi$; analogously $\hat{\mathbf{e}}_z \cdot \nabla p = \partial p/\partial z$. Substituting this into (11) and using the helical invariance of the pressure $p = p(r, u, t)$ where $u = z - \gamma\varphi$ we get

$$\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t} = -\frac{1}{\rho}\left[\frac{\partial p}{\partial \varphi} + \gamma\frac{\partial p}{\partial z}\right]$$
$$= -\frac{1}{\rho}\left[-\gamma\frac{\partial p}{\partial u} + \gamma\frac{\partial p}{\partial u}\right] = 0. \quad (12)$$

Equation (12) means that function $M_\gamma(\mathbf{x}, t)$ (8) is a material conservation law for the helically symmetric gas and fluid flows.

For the composed functions $F(M_\gamma(\mathbf{x}, t))$ we find

$$\frac{\mathrm{D}F(M_\gamma(\mathbf{x}, t))}{\mathrm{D}t} = \frac{\mathrm{d}F(M_\gamma)}{\mathrm{d}M_\gamma}\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t} = 0. \quad (13)$$

Hence, the functions $F(M_\gamma(\mathbf{x}, t))$ are the material conservation laws.[1]

Remark 2: Function $M_\gamma(\mathbf{x}, t)$ (8) does not depend on the gas density $\rho(\mathbf{x}, t)$. Therefore, $M_\gamma(\mathbf{x}, t)$ is not a linear combination of the z-projections of the angular momentum density $[\rho(\mathbf{x}, t)\mathbf{x} \times \mathbf{V}(\mathbf{x}, t)] \cdot \hat{\mathbf{e}}_z$ and the kinetic momentum density $[\rho(\mathbf{x}, t)\mathbf{V}(\mathbf{x}, t)] \cdot \hat{\mathbf{e}}_z$.

(b) For any differentiable function $G(x, y)$ of two variables x and y the composed function

$$G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t)),$$
$$M_\gamma(\mathbf{x}, t) = \left[\mathbf{x} \times \mathbf{V}(\mathbf{x}, t) + \gamma\mathbf{V}(\mathbf{x}, t)\right] \cdot \hat{\mathbf{e}}_z \quad (14)$$

is a material conservation law of an incompressible fluid flow with variable density $\rho(\mathbf{x}, t)$.

Indeed, for the incompressible fluid flow we find from the third of equation (7) that $\mathrm{D}\rho(\mathbf{x}, t)/\mathrm{D}t = 0$ that means the fluid density $\rho(\mathbf{x}, t)$ is constant along the fluid streaklines. We proved in the statement *(a)* that $\mathrm{D}M_\gamma(\mathbf{x}, t)/\mathrm{D}t = 0$, see (12). Therefore, the material derivative of the composed function $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ is:

$$\frac{\mathrm{D}G(\rho, M_\gamma)}{\mathrm{D}t} = \frac{\partial G(\rho, M_\gamma)}{\partial \rho}\frac{\mathrm{D}\rho(\mathbf{x}, t)}{\mathrm{D}t}$$
$$+ \frac{\partial G(\rho, M_\gamma)}{\partial M_\gamma}\frac{\mathrm{D}M_\gamma(\mathbf{x}, t)}{\mathrm{D}t} = 0. \quad (15)$$

Hence, all functions $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ are the material conservation laws.

Remark 3: The statements *(a)* and *(b)* imply that the diffeomorphisms defined by the helically symmetric gas and fluid flows preserve the values of function $M_\gamma(\mathbf{x}, t)$ (8), (14). Hence, the level sets $\Gamma^2_\mu(t) : M_\gamma(\mathbf{x}, t) = \mu$ for any constant μ are transported with the helically symmetric gas and fluid flows.

Remark 4: For the special values of $\mu = \mu_k$ the level sets $\Gamma^2_{\mu_k}(t)$ can be not two-dimensional surfaces but one-dimensional helically invariant curves – helices which are transported with the gas or fluid flow. These helices satisfy the equations

$$M_\gamma(\mathbf{x}, t) = \mu_k, \qquad \nabla M_\gamma(\mathbf{x}, t) = 0. \quad (16)$$

1 In the proof of the statement (a) we use only (4) and the helical invariance of the pressure $p = p(r, u, t)$. The continuity equation (5) and the entropy equation (6) are not used as well as the baroclinic equation of state $p = p(s, \rho)$. Therefore the presented proof yields that the same material conservation laws (8) exist also for the helically symmetric dynamics of the inviscid gas and incompressible fluid with possible mass decay due to the radioactive processes and with possible heat transfer due to the radiation of energy.

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

The values μ_k can be local maximum, or minimum or saddle values of function $M_\gamma(\mathbf{x}, t)$. For the saddle values μ_s the complete level sets $\Gamma_{\mu_s} : M_\gamma(\mathbf{x}, t) = \mu_s$ are two-dimensional separatrix surfaces. We call the values μ_k $(k = 1, \dots, n)$ defined by (16) the functional invariants of the gas or fluid flows.

Remark 5: Equation (6) means that the density of entropy $s(\mathbf{x}, t)$ is a material conservation law for the adiabatic gas flows. Hence, we get that for any differentiable function $G(x, y)$ of two variables x and y the composed function

$$G(s(\mathbf{x}, t), M_\gamma(\mathbf{x}, t)) \tag{17}$$

is a material conservation law for the helically symmetric adiabatic gas flows. The proof is the same as for the statement (*b*), see (15).

3 Conserved Quantities for Helically Symmetric Gas and Fluid Flows

For the helically symmetric velocity vector field $\mathbf{V}(\mathbf{x}, t)$ the function $M_\gamma(\mathbf{x}, t) = \left[\mathbf{x} \times \mathbf{V}(\mathbf{x}, t) + \gamma \mathbf{V}(\mathbf{x}, t)\right] \cdot \hat{\mathbf{e}}_z$ (8) is invariant under the Lie group of helical transformations (1). Therefore, its level sets

$$\Gamma^2_\mu(t): \qquad M_\gamma(\mathbf{x}, t) = \mu \tag{18}$$

also are invariant under the group (1) and hence under its infinite discrete subgroup of translations (2). Therefore, the surfaces $\Gamma^2_\mu(t)$ (18) always are non-compact and go to infinity in variable z. Assume that for $t = \text{const}$ the intersection of the set $\Gamma^2_\mu(t)$ with plane $z = 0$ is a union of several closed curves $C^1_\mu(t)$ which bound a compact set $D^2_\mu(t)$. The set $D^2_\mu(t)$ consists of all points $(x, y, z = 0)$ satisfying equation $M_\gamma(\mathbf{x}, t) \le \mu$.

(c) For any helically symmetric flow of inviscid compressible gas and any differentiable function $F(x)$ there exists an infinite family of conserved quantities

$$\Psi_F(\mu) = \int\limits_{D^2_\mu(t)} \rho(\mathbf{x}, t) F(M_\gamma(\mathbf{x}, t)) \mathrm{d}s, \tag{19}$$

which are functions of the real parameter μ. Here $\mathbf{x} \in D^2_\mu(t)$ and $\mathrm{d}s$ is the element of area in the plane $z = 0$.

Let the surface $\mathcal{C}^2_\mu(t) \subset \Gamma^2_\mu(t)$ be the image of the curve $C^1_\mu(t)$ under the action of the Lie group of helical transformations (1) and $\mathcal{D}^3_\mu(t)$ be the image of the compact set $D^2_\mu(t)$ under the action of the Lie group (1). Both the surface $\mathcal{C}^2_\mu(t)$ and the domain $\mathcal{D}^3_\mu(t)$ are invariant under the discrete subgroup of translations T_N (2) and hence are infinite. Surface $\mathcal{C}^2_\mu(t)$ is the boundary of the three-dimensional domain $\mathcal{D}^3_\mu(t)$. Intersection of domain $\mathcal{D}^3_\mu(t)$ with plane $z = 0$ is the two-dimensional set $D^2_\mu(t)$ bounded by the curve $C^1_\mu(t)$.

The proven in Section 2 existence of the material conservation law $M_\gamma(\mathbf{x}, t)$ (8) for the helically symmetric gas and fluid flows implies that the surface $\mathcal{C}^2_\mu(t_1)$ and the domain $\mathcal{D}^3_\mu(t_1)$ are transformed by the gas or fluid flow at time $t > t_1$ into the surface $\mathcal{C}^2_\mu(t)$ and domain $\mathcal{D}^3_\mu(t)$. Hence, the surface $\mathcal{C}^2_\mu(t)$ (on which $M_\gamma(\mathbf{x}, t) = \mu$) and its interior domain $\mathcal{D}^3_\mu(t)$ are frozen into the gas and fluid flows.

Consider the compact domain $\mathcal{D}^3_{\mu.N}(t) \subset \mathcal{D}^3_\mu(t)$ that is the part of domain $\mathcal{D}^3_\mu(t)$ between the two planes $z = 0$ and $z = 2\pi\gamma N$. The domain $\mathcal{D}^3_{\mu.N}(t)$ is bounded by a part of surface $\mathcal{C}^2_\mu(t)$ and by the two-dimensional sets $D^2_\mu(t)$ at $z = 0$ and $T_N(D^2_\mu(t))$ at $z = 2\pi\gamma N$. Here T_N is the translation (2): $z \to z + 2\pi\gamma N$. The helical transformations H_τ (1) preserve the area in the planes $z = C$ and $z = C + \gamma\tau$. Therefore using the helical invariance of $\rho(\mathbf{x}, t)$ and $M_\gamma(\mathbf{x}, t)$ we get

$$\int\limits_{\mathcal{D}^3_{\mu.N}(t_1)} \rho(\mathbf{x}, t_1) F(M_\gamma(\mathbf{x}, t_1)) \mathrm{d}x = 2\pi\gamma N \int\limits_{D^2_\mu(t_1)} \rho(\mathbf{x}, t_1) F(M_\gamma(\mathbf{x}, t_1)) \mathrm{d}s. \tag{20}$$

Let $F_{t_1.t}$ be the diffeomorphism of $\mathbb{R}^3$ defined by the compressible gas or incompressible fluid flow between the time t_1 and $t > t_1$. The diffeomorphism $F_{t_1.t}$ transforms domain $\mathcal{D}^3_{\mu.N}(t_1)$ into domain $F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t))$. Using the material conservation law (8) and conservation of mass $\rho(\mathbf{x}, t)\mathrm{d}x$ we get

$$\int\limits_{F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t_1))} \rho(\mathbf{x}, t) F(M_\gamma(\mathbf{x}, t)) \mathrm{d}x = \int\limits_{\mathcal{D}^3_{\mu.N}(t_1)} \rho(\mathbf{x}, t_1) F(M_\gamma(\mathbf{x}, t_1)) \mathrm{d}x. \tag{21}$$

Domain $F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t_1))$ belongs to the domain $\mathcal{D}^3_\mu(t)$ and is bounded by a part of surface $\mathcal{C}^2_\mu(t)$ and by the two-dimensional compact surfaces

$$\mathcal{S}_1 = F_{t_1.t}(D^2_\mu(t_1)), \qquad \mathcal{S}_N = T_N[F_{t_1.t}(D^2_\mu(t_1))]. \tag{22}$$

Let z_1 and z_N be correspondently the maximum of coordinate z on the surface $\mathcal{S}_1$ and the minimum of z on

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

the surface $\mathcal{S}_N$. Let K be the minimal integer satisfying condition $2\pi\gamma K \geq z_1$ and L be the maximal integer satisfying condition $2\pi\gamma L \leq z_N$. Invariance of the gas flow with respect to all translations T_N (2) implies that for sufficiently large N we have $L > K$ and

$$\frac{(L-K)}{N} \to 1 \quad \text{as} \quad N \to \infty. \tag{23}$$

Let the set $\mathcal{D}^3_{\mu.K.L}(t)$ be the part of the set $F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t_1))$ between the two planes $z = 2\pi\gamma K$ and $z = 2\pi\gamma L$. It is bounded by a part of surface $\mathcal{C}^2_\mu(t)$ and the two sets: $T_K(D^2_\mu(t))$ $(z = 2\pi\gamma K)$ and $T_L(D^2_\mu(t))$ $(z = 2\pi\gamma L)$. By the construction of the set $\mathcal{D}^3_{\mu.K.L}(t)$ and formula (20) for time t we have

$$\int\limits_{\mathcal{D}^3_{\mu.K.L}(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}x = 2\pi\gamma(L-K)\int\limits_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s. \tag{24}$$

Equation (23) implies that for $N \to \infty$ domain $\mathcal{D}^3_{\mu.K.L}(t)$ becomes the dominating part of the domain $F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t_1))$. Therefore, we get

$$Z = \frac{\int_{\mathcal{D}^3_{\mu.K.L}(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}x}{\int_{F_{t_1.t}(\mathcal{D}^3_{\mu.N}(t_1))} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}x} \to 1 \quad \text{as } N \to \infty. \tag{25}$$

Substituting into (25) formulas (24) and (21) and using (20) we find

$$Z = \frac{2\pi\gamma(L-K)\int_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s}{\int_{\mathcal{D}^3_{\mu.N}(t_1)} \rho(\mathbf{x}, t_1)F(M_\gamma(\mathbf{x}, t_1))\mathrm{d}x} = \frac{(L-K)}{N}\frac{\int_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s}{\int_{D^2_\mu(t_1)} \rho(\mathbf{x}, t_1)F(M_\gamma(\mathbf{x}, t_1))\mathrm{d}s}. \tag{26}$$

Equations (23) and (25) show that $(L-K)/N \to 1$ and $Z \to 1$ as $N \to \infty$. Therefore, from (26) we get

$$\frac{\int_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s}{\int_{D^2_\mu(t_1)} \rho(\mathbf{x}, t_1)F(M_\gamma(\mathbf{x}, t_1))\mathrm{d}s} = Z\frac{N}{L-K} \to 1 \tag{27}$$

as $N \to \infty$. The left hand side of (27) does not depend on N. Therefore it is equal to its limit at $N \to \infty$. Hence, we find from (27):

$$\frac{\int_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s}{\int_{D^2_\mu(t_1)} \rho(\mathbf{x}, t_1)F(M_\gamma(\mathbf{x}, t_1))\mathrm{d}s} = 1.$$

This equation means that function $\Psi_F(\mu) = \int_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s$ (19) does not depend on time t and, therefore, for each μ is a conserved quantity for the helically symmetric gas flows.

Remark 6: Intersection of the helically invariant set $\Gamma^2_\mu(t) : M_\gamma(\mathbf{x}, t) = \mu$ with a plane $z = C = \gamma\tau$ contains the closed curve $H_\tau(C^1_\mu(t))$ that bounds the compact domain $H_\tau(D^2_\mu(t))$. The helical transformations H_τ (1) preserve the area in the planes $z = 0$ and $z = C = \gamma\tau$. Therefore using the helical invariance of $\rho(\mathbf{x}, t)$ and $M_\gamma(\mathbf{x}, t)$ we find

$$\int\limits_{D^2_\mu(t)} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s = \int\limits_{H_\tau(D^2_\mu(t))} \rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))\mathrm{d}s. \tag{28}$$

The equality (28) implies that the conserved quantities $\Psi_F(\mu)$ calculated for $z = 0$ and for $z = C = \gamma\tau$ coincide and do not depend on the constant value of $z = C$.

(d) For any helically symmetric incompressible fluid flow with variable density $\rho(\mathbf{x}, t)$ and any differentiable function $G(x, y)$ of two variables x and y there exists an infinite family of conserved quantities

$$\Psi_G(\mu) = \int\limits_{D^2_\mu(t)} G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))\mathrm{d}s, \tag{29}$$

which are functions of the real parameter μ. Here $\mathbf{x} \in D^2_\mu$ and $\mathrm{d}s$ is the area element in the plane $z = 0$.

The proof follows the same way as for the statement *(c)*, by substituting $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ instead of $\rho(\mathbf{x}, t)F(M_\gamma(\mathbf{x}, t))$ and using the result of statement (b) that incompressible fluid flows preserve the three-dimensional volume $\mathrm{d}x$ and have two functionally independent material conservation laws $\rho(\mathbf{x}, t)$ and $M_\gamma(\mathbf{x}, t)$.

Remark 7: For the adiabatic flows of compressible gas we proved that functions $G(s(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ (17) are material conservation laws. Therefore, using the same arguments as in the statements *(c)* and *(d)* we get that functions

$$\Psi_G(\mu) = \int\limits_{D^2_\mu(t)} \rho(\mathbf{x}, t)G(s(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))\mathrm{d}s$$

for arbitrary differentiable functions $G(x, y)$ are conserved quantities for the helically symmetric adiabatic gas flows.

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

4 An Example

Consider an ideal incompressible fluid flow with velocity $\mathbf{V}(\mathbf{x}, t)$ (3) having at $t = 0$ the form

$$\mathbf{V}(\mathbf{x}) = f(r, u)\hat{\mathbf{e}}_r + r\exp(-r^2)\cos(z/\gamma - \varphi)\hat{\mathbf{e}}_z + r\exp(-r^2)\hat{\mathbf{e}}_\varphi, \quad (30)$$

$$f(r, u) = \frac{1}{2\gamma}\left(\frac{1}{r}\int_0^r e^{-x^2}\mathrm{d}x - e^{-r^2}\right)\sin(z/\gamma - \varphi), \quad (31)$$

where $u = z - \gamma\varphi$. Vector field $\mathbf{V}(\mathbf{x})$ (30), (31) satisfies the incompressibility equation $\nabla \cdot \mathbf{V} = 0$ and is smooth everywhere in $\mathbb{R}^3$.

Vector field $\mathbf{V}(\mathbf{x})$ (30) evidently is invariant under the Lie group of helical transformations (1). Using formula $\mathbf{x} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$ we find $[\mathbf{x} \times \mathbf{V}(\mathbf{x})] \cdot \hat{\mathbf{e}}_z = [\hat{\mathbf{e}}_z \times (r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)] \cdot \mathbf{V}(\mathbf{x}) = r^2\exp(-r^2)$. Therefore, the material conservation law $M_\gamma(\mathbf{x}, t)$ (8) at $t = 0$ takes the form

$$M_\gamma(\mathbf{x}) = \left[\mathbf{x} \times \mathbf{V}(\mathbf{x}) + \gamma\mathbf{V}(\mathbf{x})\right] \cdot \hat{\mathbf{e}}_z = \exp(-r^2)\left[r^2 + \gamma r\cos(z/\gamma - \varphi)\right]. \quad (32)$$

Function $M_\gamma(\mathbf{x})$ (32) is invariant under the helical transformations (1). On the plane $z = 0$ function $M_\gamma(\mathbf{x})$ takes the form in the Cartesian coordinates x, y:

$$M_\gamma(r, \varphi) = \exp(-r^2)\left[r^2 + \gamma r\cos(\varphi)\right] = \exp(-x^2 - y^2)(x^2 + y^2 + \gamma x).$$

The level curves of this function for $\gamma = \sqrt{3}/2$ are shown in Figure 1: they all are either smooth closed curves or three points c_1, c_2, s_1 of extrema of function $M_{\sqrt{3}/2}(r, \varphi)$ or two separatrices of the saddle point s_1. The range of function $M_{\sqrt{3}/2}(r, \varphi)$ is the segment $[\mu_1, \mu_2]$, where the minimal value $\mu_1 = -((\sqrt{3} - 1)/4)\exp(-1 + \sqrt{3}/2) \approx -0.1601$ is achieved at the point c_1: $(x = -(\sqrt{3} - 1)/2, y = 0)$ and the maximal value $\mu_2 = (3/2)\exp(-3/4) \approx 0.7085$ is achieved at the point c_2: $(x = \sqrt{3}/2, y = 0)$. Function $M_{\sqrt{3}/2}(r, \varphi)$ has one saddle point s_1: $(x = -(\sqrt{3} + 1)/2, y = 0)$ where $M_{\sqrt{3}/2}(s_1) = \mu_3 = ((\sqrt{3} + 1)/4)\exp(-1 - \sqrt{3}/2) \approx 0.1057$.

Remark 8: The values

$$M_\gamma(c_1) = \mu_1 \approx -0.1601, \quad M_\gamma(c_2) = \mu_2 \approx 0.7085, \quad M_\gamma(s_1) = \mu_3 \approx 0.1057$$

are the functional invariants (16) for the fluid flow (30), (31) with $\gamma = \sqrt{3}/2$.

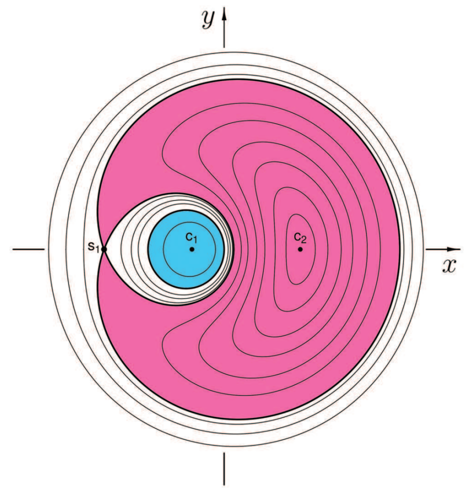

Figure 1: Level curves of the material conservation law $M_\gamma(r, u) = \exp(-r^2)\left[r^2 + \gamma r\cos(z/\gamma - \varphi)\right]$ at $t = 0$ on the plane $z = 0$. Here $\gamma = \sqrt{3}/2$.

Function $M_{\sqrt{3}/2}(r, \varphi)$ is negative in the disk $(x + \sqrt{3}/4)^2 + y^2 \leq 3/16$ (shown in blue colour in Fig. 1). The domain where $\mu_3 \leq M_{\sqrt{3}/2}(r, \varphi) \leq \mu_2$ is shown in pink colour in Figure 1. In the (infinite) white domain we have $0 < M_{\sqrt{3}/2}(r, \varphi) < \mu_3$.

Let D^2_μ be the set of all points (r, ϕ) where $M_{\sqrt{3}/2}(r, \varphi) \leq \mu$. The set $D^2_{\mu_1}$ consists of the single point c_1; the set $D^2_{\mu_2}$ is the whole plane $\mathbb{R}^2$. Assume that the fluid density $\rho = 1$ and consider two functions of parameter μ:

$$\Psi_1(\mu) = \int_{D^2_\mu} M_{\sqrt{3}/2}(r, \varphi)r\mathrm{d}r\mathrm{d}\varphi,$$

$$\Psi_2(\mu) = \int_{D^2_\mu} \left(M_{\sqrt{3}/2}(r, \varphi)\right)^2 r\mathrm{d}r\mathrm{d}\varphi. \quad (33)$$

Functions $\Psi_1(\mu)$, $\Psi_2(\mu)$ (33) are special cases of functions $\Psi_G(\mu)$ (29) for which $\rho = 1$ and respectively $G(\rho, M_\gamma) = M_\gamma$ and $G(\rho, M_\gamma) = (M_\gamma)^2$.

It is sufficient to consider functions $\Psi_1(\mu)$ and $\Psi_2(\mu)$ only for $\mu \in [\mu_1, \mu_2]$, because for all $\mu \geq \mu_2$ we have $\Psi_j(\mu) = \Psi_j(\mu_2)$ and for all $\mu \leq \mu_1$ we have $\Psi_j(\mu) = \Psi_j(\mu_1) = 0$, $j = 1, 2$. The values of functions $\Psi_1(\mu)$ and $\Psi_2(\mu)$ at the point μ_2 are the integrals (33) over the whole plane $\mathbb{R}^2$. After the integrations we get: $\Psi_1(\mu_2) = \pi$, $\Psi_2(\mu_2) = \frac{11}{32}\pi$. The plots of functions (33) are shown in Figure 2. Functions $\Psi_1(\mu)$ and $\Psi_2(\mu)$ (33) are invariants of the helically symmetric dynamics of an ideal incompressible fluid with initial (at $t = 0$) velocity $\mathbf{V}(\mathbf{x})$ of the form (30), (31) with $\gamma = \sqrt{3}/2$.

During the helically symmetric dynamics of fluid the phase portrait of the level sets $M_{\sqrt{3}/2}(r, \varphi, t) = \mu$ does depend on time t but remains topologically equivalent to

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM

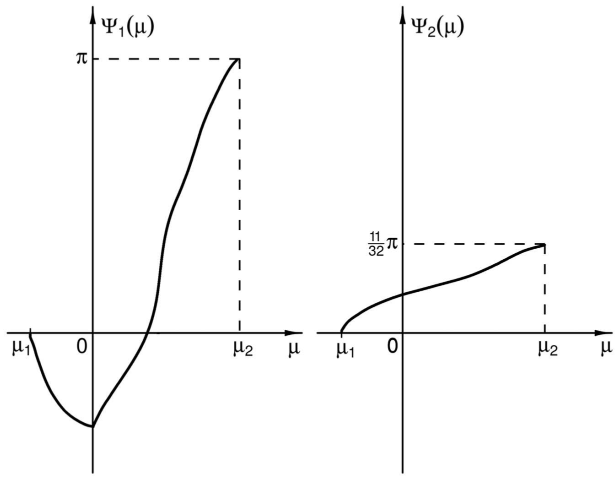

Figure 2: Plots of invariant functions $\Psi_1(\mu)$ and $\Psi_2(\mu)$ for the incompressible fluid flow with velocity (30), (31), $\gamma = \sqrt{3}/2$.

that in Figure 1. The plots of the corresponding functions $\Psi_1(\mu)$ and $\Psi_2(\mu)$ in Figure 2 do not depend on time t and therefore are new invariants of fluid dynamics.

5 Conclusion

The following main results were obtained in this work:

- We derived new material conservation laws $F(M_\gamma(\mathbf{x}, t))$ (8) and functional invariants μ_k (16) for the helically symmetric flows of inviscid compressible gas satisfying Euler's equation (4).
- We obtained for the helically symmetric flows of inviscid incompressible fluid with a variable density $\rho(\mathbf{x}, t)$ the material conservation laws $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$ (14). These contain an arbitrary differentiable function $G(x, y)$ of two variables x and y and generalize the laws $F(\zeta)$ (where $F(\zeta)$ is an arbitrary function of one variable) derived by Kelbin et al. in [5] for the incompressible fluid flows with constant density ρ.
- We introduced new conserved quantities $\Psi_F(\mu)$ (19) and $\Psi_G(\mu)$ (29) for the helically symmetric flows of inviscid compressible gas and an ideal incompressible fluid which are integrals over special compact two-dimensional sets D^2_μ of the products of density $\rho(\mathbf{x}, t)$ with the material conservation laws $F(M_\gamma(\mathbf{x}, t))$ and $G(\rho(\mathbf{x}, t), M_\gamma(\mathbf{x}, t))$.
- The new invariants $\Psi_F(\mu)$ (19) and $\Psi_G(\mu)$ (29) are piecewise differentiable functions of the real parameter μ. If two helically symmetric states of inviscid compressible gas or incompressible fluid are dynamically connected then the corresponding to them functions $\Psi_F(\mu)$ (19) and $\Psi_G(\mu)$ (29) must coincide.

References

[1] H. Lamb, in: Hydrodynamics, Cambridge University Press, Cambridge 1879.
[2] G. K. Batchelor, in: An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge 1967.
[3] H. K. Moffatt, Phil. Trans. R. Soc. Lond. A **333**, 321 (1990).
[4] W. M. Hicks, Phil. Trans. R. Soc. Lond. A **192**, 33 (1899).
[5] O. Kelbin, A. F. Cheviakov, and M. Oberlack, J. Fluid Mech. **721**, 340 (2013).
[6] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Phys. Fluids **1**, 281 (1958).
[7] O. I. Bogoyavlenskij, Phys. Rev. E **62**, 8616 (2000).
[8] R. Rotunno, J. Atmos. Sci. **36**, 140 (1979).
[9] C. R. Church, J. T. Snow, G. L. Baker, and E. M. Agee, J. Atmos. Sci. **36**, 1755 (1979).
[10] B. H. Fiedler and R. Rotunno, J. Atmos. Sci. **43**, 2328 (1986).
[11] A. F. Cheviakov and M. Oberlack, J. Fluid Mech. **760**, 368 (2014).
[12] O. Bogoyavlenskij, Z. Naturforsch. A, **73**, 931 (2018).
[13] D. Serre, Physica **13D**, 105 (1984).
[14] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[15] O. Bogoyavlenskij, Phys. Rev. E **95**, 043104, 14 pp (2017).
[16] O. Bogoyavlenskij, J. Math. Phys. **58**, 013101 (2017).
[17] O. Bogoyavlenskij, J. Math. Anal. Appl. **450**, 21 (2017).
[18] H. K. Moffatt, Corrigendum. J. Fluid Mech. **830**, 821 (2017).
[19] W. Thomson (Lord Kelvin), Trans. R. Soc. Edin. **25**, 217 (1869).
[20] W. Thomson (Lord Kelvin), Proc. R. Soc. Edin. **14**, 115 (1875).

Brought to you by | Queens University Library (Ontario Canada)
Authenticated
Download Date | 4/12/19 4:13 PM