Physics of Fluids ARTICLE scitation.org/journal/phf

Safety factor for time-dependent axisymmetric flows of barotropic gas and ideal incompressible fluid

Cite as: Phys. Fluids 31, 026106 (2019); doi: 10.1063/1.5083630
Submitted: 29 November 2018 • Accepted: 20 January 2019 •
Published Online: 13 February 2019

View Online Export Citation CrossMark

Oleg Bogoyavlenskij

AFFILIATIONS

Department of Mathematics, Queen's University, Kingston, Ontario K7L 3N6, Canada

ABSTRACT

The safety factor $q(r, z, t)$ is proved to be a material conservation law for the time-dependent axisymmetric barotropic compressible gas flows and ideal incompressible fluid flows with constant density ρ. Infinite families of conserved quantities connected with the safety factor are derived. The existence of maximal vortex rings and vortex blobs which are frozen into the axisymmetric inviscid gas and fluid flows is demonstrated. A stratification in the space of ideal gas and fluid flows is obtained: if two axisymmetric states of the barotropic gas or fluid with constant density ρ are dynamically connected, then their total numbers of vortex rings must be equal (the same for the total numbers of vortex blobs) and the infinitely many corresponding conserved quantities must coincide.

I. INTRODUCTION

We study Euler's equations of the inviscid compressible gas mechanics[1]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nabla\Phi, \quad \frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{V}) = 0, \tag{1.1}$$

where $\mathbf{V}(\mathbf{x}, t)$ is the fluid velocity of class C^2, $p(\mathbf{x}, t)$ is the pressure, $\rho(\mathbf{x}, t)$ is the gas density, and $\Phi(\mathbf{x}, t)$ is the Newtonian gravitational potential. Equation of state is assumed to be barotropic: $p(\mathbf{x}, t) = f(\rho(\mathbf{x}, t))$, where $f(u)$ is a differentiable function with $df(u)/du > 0$. For example, gas can be viewed as barotropic with $p(\mathbf{x}, t) = A\rho^\gamma(\mathbf{x}, t)$, where A and γ are constants, $\gamma \geq 1$.

As known,[2,3] Euler's equations (1.1) for the barotropic compressible gas flows imply that the vector field

$$\mathbf{\Omega}(\mathbf{x}, t) = \frac{\nabla\times\mathbf{V}(\mathbf{x}, t)}{\rho(\mathbf{x}, t)} \tag{1.2}$$

satisfies the equation

$$\frac{\partial \mathbf{\Omega}}{\partial t} = (\mathbf{\Omega}\cdot\nabla)\mathbf{V} - (\mathbf{V}\cdot\nabla)\mathbf{\Omega}. \tag{1.3}$$

Equation (1.3) yields[2,3] that the vector field $\mathbf{\Omega}(\mathbf{x}, t)$ (1.2) is transported in time by the gas flow diffeomorphisms (or is "frozen" into the compressible gas flow).[31] This implies that any closed trajectory of the vector field $\mathbf{\Omega}$ (1.2) for time t_1 [that is called a "vortex knot" because vector field $\mathbf{\Omega}$ (1.2) is proportional to the vorticity field $\nabla\times\mathbf{V}$] is transformed by the gas flow at a time $t > t_1$ into an isotopic vortex knot. Therefore the frozenness of the vector field $\mathbf{\Omega}(\mathbf{x}, t)$ (1.2) into the barotropic gas flow leads to the preservation in time of the discrete topological invariants of vortex knots and the integer valued Gauss linking number for any two closed and linked vorticity field lines. These facts are true for the inviscid barotropic gas flows and are well-known for the ideal incompressible fluid flows with constant density ρ;[4] however, they are not true for the viscous gas and fluid flows.

The z-axisymmetric velocity field $\mathbf{V}(\mathbf{x}, t)$ in the cylindrical coordinates r, z, φ has the form

$$\mathbf{V}(r, z, t) = u(r, z, t)\hat{\mathbf{e}}_r + v(r, z, t)\hat{\mathbf{e}}_z + w(r, z, t)\hat{\mathbf{e}}_\varphi, \tag{1.4}$$

where $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_z, \hat{\mathbf{e}}_\varphi$ are vectors of unit length in directions of coordinates r, z, φ. The corresponding vorticity vector field is

$$\nabla\times\mathbf{V} = -\frac{(rw)_z}{r}\hat{\mathbf{e}}_r + \frac{(rw)_r}{r}\hat{\mathbf{e}}_z + (u_z - v_r)\hat{\mathbf{e}}_\varphi. \tag{1.5}$$

Therefore the vector field $\mathbf{\Omega}(\mathbf{x}, t)$ (1.2) (that is frozen into the barotropic compressible gas flow) takes the form

$$\mathbf{\Omega}(r,z,t) = -\frac{(rw)_z}{\rho r}\hat{\mathbf{e}}_r + \frac{(rw)_r}{\rho r}\hat{\mathbf{e}}_z + \frac{u_z - v_r}{\rho}\hat{\mathbf{e}}_\varphi. \tag{1.6}$$

We study also Euler's equations of the ideal incompressible fluid mechanics

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nabla\Phi, \quad \nabla\cdot\mathbf{V} = 0, \tag{1.7}$$

where the fluid density ρ is supposed to be constant. For the case of incompressible fluid, Eq. (1.3) is true if ρ = const and implies that the vorticity vector field $\nabla\times\mathbf{V} = \rho\mathbf{\Omega}$ is frozen into the fluid flow[1,3] and hence all topological invariants of vortex knots are preserved along the incompressible fluid flows.[4]

In our paper,[5] we proved that functions $rw(r, z, t)$ and $F(rw(r, z, t))$ are material conservation laws for the inviscid compressible gas flows with arbitrary equation of state [here $F(u)$ is an arbitrary differentiable function of u].

Remark 1. Function $rw(r, z, t)$ has the form

$$rw(r,z,t) = (\mathbf{x}\times\mathbf{V}(\mathbf{x},t))\cdot\hat{\mathbf{e}}_z. \tag{1.8}$$

The z-projection of the density of gas angular momentum is $\mathcal{P}(\mathbf{x},t) = \rho(\mathbf{x},t)(\mathbf{x}\times\mathbf{V}(\mathbf{x},t))\cdot\hat{\mathbf{e}}_z$. Therefore function $rw(r, z, t)$ (1.8) is different from the z-projection of the density of gas angular momentum $\mathcal{P}(\mathbf{x},t)$. The latter is not a material conservation law.

Remark 2. The material conservation law (1.8) is invariant under the transformations

$$\mathbf{V}(\mathbf{x},t) \longrightarrow \mathbf{V}(\mathbf{x},t) + U\hat{\mathbf{e}}_z, \tag{1.9}$$

where U is an arbitrary constant. This invariance is connected with the special form of the Galilean invariance of Euler's equations (1.1) for the z-axisymmetric flows.

For the z-axisymmetric ideal incompressible fluid flows with constant density ρ, Hicks had shown in Ref. 6 on p. 97 that the function $rw(r, z, t)$ satisfies the equation

$$\frac{\partial(rw)}{\partial t} + u\frac{\partial(rw)}{\partial r} + v\frac{\partial(rw)}{\partial z} = 0. \tag{1.10}$$

In modern terminology, Eq. (1.10) means that the function $rw(r, z, t)$ is a material conservation law since it is conserved along the fluid streaklines. Batchelor had presented in Ref. 1 on p. 544 the equation "$D(rw)/Dt = 0$ (7·5·7)" that is equivalent to Eq. (1.10). Kelbin, Cheviakov, and Oberlack presented in Ref. 7 on p. 351 the equation $d(ru^\varphi)/dt = 0$ that coincides with Eq. (1.10) and derived from it for the axisymmetric fluid flows with constant density ρ the material conservation laws $F(rw(r, z, t))$ [where $F(x)$ is an arbitrary differentiable function of x]. In Ref. 5, we demonstrated that for the z-axisymmetric flows of incompressible fluid with variable density $\rho(r, z, t)$ the functions $G(\rho(r, z, t), rw(r, z, t))$ are material conservation laws [here $G(x, y)$ is an arbitrary differentiable function of x, y].

We apply the results of Ref. 5 to construct in Sec. II the maximal vortex rings and vortex blobs for the axisymmetric flows of the inviscid compressible gas and ideal incompressible fluid.[32]

The time-independent safety factor $q(r, z)$ was studied before only for steady axisymmetric magnetic fields and for steady axisymmetric flows of ideal incompressible fluid. The magnetic safety factor for the z-axisymmetric steady magnetic fields $\mathbf{B}(r, z)$ is widely used in plasma physics.[11,14–19] For the steady flows of ideal incompressible fluid with constant density ρ, the safety factor for the vorticity vector field $\nabla\times\mathbf{V}$ (1.5) was studied in Refs. 4, 6, 12, and 13 where it was called a "pitch."

Remark 3. To the best of our knowledge, this work is the first where the safety factor $q(r, z, t)$ is studied for the time-dependent barotropic flows of compressible gas and incompressible fluid. The time-dependent magnetic safety factor $q_m(r, z, t)$ for the z-axisymmetric magnetic fields $\mathbf{B}(r, z, t)$ was studied in our paper.[20]

In Sec. IV, we define the time-dependent safety factor $q(r, z, t)$ for the unsteady vector field $\mathbf{\Omega}(r, z, t)$ [(1.2), (1.6)] for the barotropic compressible gas flows and time-dependent ideal incompressible fluid flows with constant density ρ and prove in Sec. V that the safety factor $q(r, z, t)$ is a new material conservation law [this means that $q(r, z, t)$ is preserved along the gas and fluid streaklines].

New conserved quantities and functional invariants of the axisymmetric gas and fluid flows are presented in Secs. VI and VII together with the proof of their functional independence from the helicity. An extension of the classical Kelvin's problem is presented in Sec. VIII.

II. VORTEX RINGS AND VORTEX BLOBS

For the z-axisymmetric flows of inviscid gas, we consider for any constant time t the surfaces $rw(r, z, t) = rw$ = const. Due to their z-rotational symmetry, the surfaces are either tori $T^2_{rw} = C_{rw}\times\mathbb{S}^1 \subset \mathbb{R}^3$ or spheres S^2_{rw} or cylinders $B^2_{rw} = R_{rw}\times\mathbb{S}^1$. The closed curves C_{rw} and infinite lines R_{rw} lie in the plane (r, z) (that means they are purely poloidal) and satisfy the equation $rw(r, z, t) = rw$ = const. The circle $\mathbb{S}^1$ corresponds to the angular coordinate $0 \le \varphi \le 2\pi$ and is z-axisymmetric.

The vector field $\mathbf{\Omega}(r, z, t)$ (1.6) and the vorticity field $\nabla\times\mathbf{V}(r, z, t)$ (1.5) evidently are tangent to the surfaces $rw(r, z, t) = rw$ = const.

If a surface $rw(r, z, t) = rw$ = const is a torus $T^2_{rw} = C_{rw}\times\mathbb{S}^1$, then the vorticity lines on T^2_{rw} are either infinite helical curves or closed curves.

Dynamics along the vector field $\mathbf{\Omega}(r, z, t)$ integral lines is defined by the system (t = const)

$$\frac{d\mathbf{x}}{d\tau} = \frac{dx}{d\tau}\hat{\mathbf{e}}_x + \frac{dy}{d\tau}\hat{\mathbf{e}}_y + \frac{dz}{d\tau}\hat{\mathbf{e}}_z = \frac{dr}{d\tau}\hat{\mathbf{e}}_r + r\frac{d\varphi}{d\tau}\hat{\mathbf{e}}_\varphi + \frac{dz}{d\tau}\hat{\mathbf{e}}_z = \mathbf{\Omega}(r,z,t),$$

which by virtue of Eq. (1.6) has the form

$$\frac{dr}{d\tau} = -\frac{(rw(r,z,t))_z}{\rho r}, \quad \frac{dz}{d\tau} = \frac{(rw(r,z,t))_r}{\rho r}, \tag{2.1}$$

$$\frac{d\varphi}{d\tau} = \frac{u_z - v_r}{\rho r}. \tag{2.2}$$

After the time change $\tau \to \tau_1 : d\tau_1/d\tau = 1/(\rho r)$, system (2.1) for t = const becomes an autonomous in the time τ_1 Hamiltonian system with the Hamiltonian function $H(r, z) = rw(r, z, t)$,

$$\frac{dr}{d\tau_1} = -\frac{\partial H(r,z,t)}{\partial z}, \quad \frac{dz}{d\tau_1} = \frac{\partial H(r,z,t)}{\partial r}. \tag{2.3}$$

We will use the following properties of any autonomous Hamiltonian system (2.3) in $\mathbb{R}^2$:[21] (a) its Hamiltonian function $H(r, z)$ is a first integral; (b) it preserves the area $drdz$; (c) all its non-degenerate equilibria are either saddles s_ℓ or centers c_k. These properties imply that every closed trajectory C_{rw} ($rw(r, z, t) = rw$ = const) of system (2.1) belongs to one of the poloidal sets $D_k(t) \subset \mathbb{R}^2$ satisfying the following conditions:

1. $D_k(t)$ is invariant with respect to system (2.1) and (2.3);
2. a dense open subset of $D_k(t)$ is filled with closed trajectories of system (2.1) and (2.3);
3. all trajectories of system (2.1) and (2.3) in the set $D_k(t)$ have a finite Euclidean length and are either closed curves or separatrices of the equilibrium points in $D_k(t)$;
4. the set $D_k(t)$ is connected and compact;
5. the set $D_k(t)$ is the largest among the sets satisfying conditions 1–4 and having a non-empty intersection with $D_k(t)$.

We will call each set $D_k(t)$ satisfying conditions 1–5 a maximal invariant compact set (the term "maximal" refers to condition 5).

The boundary $\partial D_k(t)$ is a 1-dimensional set $\mathcal{C}_k(t)$ that is invariant with respect to the dynamical system (2.1) and (2.3) and therefore satisfies the equation $rw(r, z, t) = b_k$ = const.

The closed trajectories $C_{rw(t)}$ of system (2.1) (t = const) define after rotation around the axis of symmetry z the 2-dimensional tori $T^2_{rw} = C_{rw} \times \mathbb{S}^1 \subset \mathbb{R}^3$ that are invariant with respect to system (2.1) and (2.2) and therefore are called $\mathbf{\Omega}(r, z, t)$-invariant. Rotation of a maximal set $D_k(t)$ around the axis z defines a 3-dimensional $\mathbf{\Omega}(r, z, t)$-invariant closed set $\mathcal{D}^3_k(t) = D_k(t) \times \mathbb{S}^1 \subset \mathbb{R}^3$. Rotation of the boundary curve $\mathcal{C}_k(t) = \partial D_k(t)$ around the axis z defines the boundary $\partial\mathcal{D}^3_k(t) = \mathcal{C}_k(t) \times \mathbb{S}^1$.

The total mass of gas inside the invariant domain $\mathcal{D}^3_k$ is

$$\text{Mass}\ \mathcal{D}^3_k(t) = \int_{\mathcal{D}^3_k(t)} \rho(r,z,t) r dr dz d\varphi = 2\pi \int_{D_k(t)} \rho(r,z,t) r dr dz. \tag{2.4}$$

Remark 4. If $\mathcal{C}_k(t)$ is a closed curve lying in the domain $r > 0$, then the closed set $\mathcal{D}^3_k(t)$ is a maximal $\mathbf{\Omega}(r, z, t)$-invariant ring $\mathcal{R}^3_k(t)$ which we call a vortex ring because it is invariant also with respect to the vorticity field $\nabla \times \mathbf{V} = \rho\mathbf{\Omega}$. It is bounded by the torus $\mathcal{T}^2_k(t) = \mathcal{C}_k(t) \times \mathbb{S}^1$. Otherwise the closed and bounded set $\mathcal{D}^3_k(t)$ is a spheroid $\mathcal{S}^3_\ell(t)$ (which we call a vortex blob) containing a segment $S_\ell(t)$ of the axis of symmetry z ($r = 0$).

Example 1. Let us consider an ideal incompressible fluid equilibrium defined by the exact formula

$$\mathbf{V}(r,z) = r^{-1}(-\psi_z \hat{\mathbf{e}}_r + \psi_r \hat{\mathbf{e}}_z + \psi \hat{\mathbf{e}}_\varphi). \tag{2.5}$$

Assume that the streamfunction $\psi(r, z)$ has the form

$$\psi(r,z) = r^2[\xi - zG_3(R)], \quad G_3(R) = \frac{1}{R^4}\left[(3 - R^2)\frac{\sin R}{R} - 3\cos R\right], \tag{2.6}$$

where $R = \sqrt{r^2 + z^2}$ and ξ is an arbitrary parameter. It is easy to verify that the function $\psi(r, z)$ (2.6) satisfies the special case of the Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} = r^2\frac{dF}{d\psi} - \mathcal{G}\frac{d\mathcal{G}}{d\psi},$$

for which the arbitrary functions $F(\psi)$ and $\mathcal{G}(\psi)$ have the form $F(\psi) = \xi\psi$ and $\mathcal{G}(\psi) = \psi$. Therefore the function $\psi(r, z)$ (2.6) defines an ideal incompressible fluid equilibrium. Figure 1 demonstrates the phase portrait of the dynamical system (2.1) with $\rho(r, z) = 1$ for the exact fluid equilibrium (2.5) that has the function $rw(r, z) = \psi(r, z) = r^2[0.01 - zG_3(R)]$, corresponding to $\xi = 0.01$ in Eq. (2.6). The plot in Fig. 1 contains four maximal

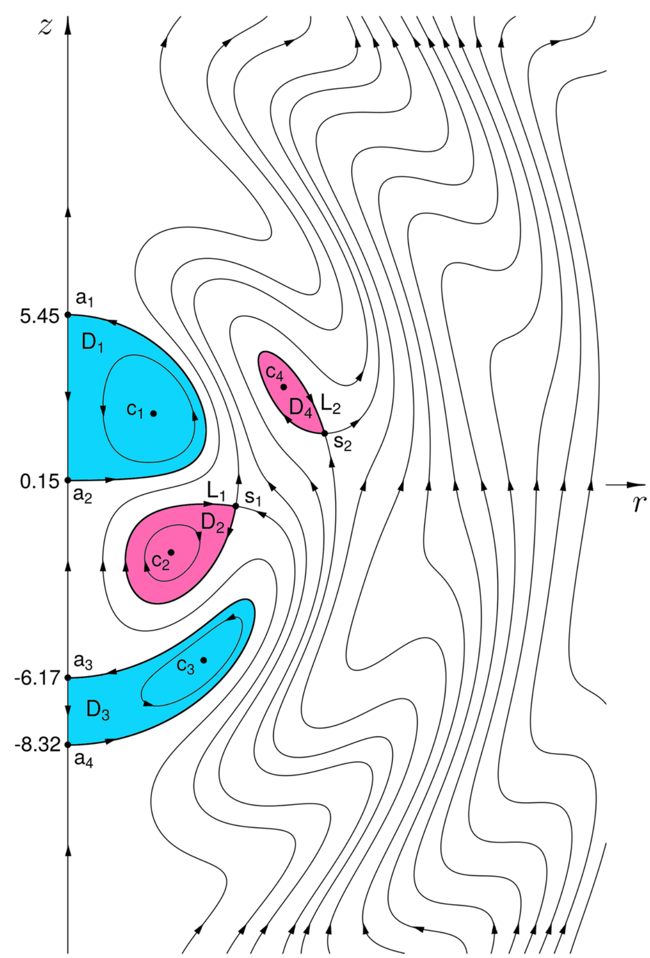

FIG. 1. Poloidal projections of vortex rings D_2 and D_4 and vortex blobs D_1 and D_3 for the exact fluid equilibrium with the streamfunction $\psi(r, z) = rw(r, z) = r^2[0.01 - zG_3(R)]$ (2.6).

invariant compact sets D_k which are densely filled with closed trajectories C_{rw} ($rw(r, z)$ = const) encircling the center equilibrium points c_k, $k = 1, 2, 3, 4$. The maximal sets D_1, D_3 (blue) contain the equilibrium points c_1 and c_3 and are bounded by the two separatrices of the saddle equilibria (a_1, a_2), (a_3, a_4), respectively. Rotation of the sets D_1, D_3 around the axis of symmetry z defines two vortex blobs $\mathcal{S}^3_k$; on their boundaries, the equation $rw(r, z) = 0$ holds. The maximal invariant compact sets D_2, D_4 (pink) contain the equilibrium points c_2 and c_4 and are bounded by the loop separatrices L_1, L_2 of the saddle equilibria s_1, s_2. Rotation of the sets D_2, D_4 around the axis z defines two vortex rings $\mathcal{R}^3_j$ that are bounded by the tori $\mathcal{T}^2_j = L_j \times \mathbb{S}^1$, $j = 1, 2$.

III. DYNAMICS IN THE POLOIDAL COORDINATES (r, z)

Suppose a z-axisymmetric vector field $\mathbf{\Omega}(r, z, t_1)$ at $t = t_1$ has a closed integral curve $L \subset \mathbb{R}^3$. Then its poloidal projection $P(L)$ (ignoring variable φ) is a closed trajectory of system (2.1) in the plane (r, z) and hence has no self-intersections and satisfies the equation $rw(r, z, t_1)$ = const. Therefore from the above, we get that there exists a maximal invariant set $D(t_1) \subset \mathbb{R}^2$ that contains the curve $P(L)$ and is densely filled with closed trajectories $C_{rw(t_1)}$.

For the z-axisymmetric flows, we get that the induced diffeomorphisms defined in the poloidal plane (r, z) by the poloidal projection of the velocity field $\mathbf{V}(r, z, t)$ preserve the axis of symmetry z $(r = 0)$ and the invariant domain $r > 0$. The frozenness of the vector field $\mathbf{\Omega}(r, z, t)$ into the barotropic gas flows[2,3] implies for the z-axisymmetric case that the poloidal projection of the vector field $\mathbf{\Omega}(r, z, t)$ is frozen into the induced dynamics of fluid in the poloidal coordinates (r, z). Therefore the phase portrait of dynamical system (2.1) is changed in time t by the induced diffeomorphisms of the poloidal plane (r, z).

Let $F_{t_1 \cdot t}$ be the induced diffeomorphism defined by the fluid dynamics from time t_1 to time $t > t_1$. Any closed trajectory $C_{rw(t_1)}$ of system (2.1) at time t_1 in the invariant domain $r > 0$ is transformed by the diffeomorphism $F_{t_1 \cdot t}$ into another closed trajectory $C_{rw(t)}$ of system (2.1) at time t in the domain $r > 0$. The corresponding constant values of the function $rw(r, z, t)$ are the same because the function $rw(r, z, t)$ is the material conservation law.[5] Since any maximal invariant compact set $D_k(t_1)$ [that is densely filled with the closed trajectories $C_{rw(t_1)}$] is transformed by the diffeomorphism $F_{t_1 \cdot t}$ into the maximal set $D_k(t)$, the frozenness of the vector field $\mathbf{\Omega}(r, z, t)$ into the fluid flow implies that the image $F_{t_1 \cdot t}(D_k(t_1))$ coincides with the set $D_k(t)$.

Any compressible gas flow preserves the total mass of gas inside any domain transported with the flow; this follows from the second of Eqs. (1.1). Hence we get the conservation of mass (2.4),

$$\text{Mass } \mathcal{D}^3_k(t) = \text{Mass } \mathcal{D}^3_k(t_1). \tag{3.1}$$

Since the induced diffeomorphisms $F_{t_1 \cdot t}$ preserve the axis of symmetry z $(r = 0)$ and the domain $r > 0$, we get that the total number N_r of vortex rings $\mathcal{R}^3_k(t) = D_k(t) \times \mathbb{S}^1$ and their masses (3.1) are constant in time. The same is true for the total number N_s of closed vortex blobs $\mathcal{S}^3_\ell(t) = D_\ell(t) \times \mathbb{S}^1$ and their masses (3.1). No collapses or touching between vortex rings or blobs can occur during the axisymmetric ideal gas or fluid dynamics. The above results mean that the vortex rings $\mathcal{R}^3_k(t)$ and vortex blobs $\mathcal{S}^3_\ell(t)$ are transported with the fluid flow or are "frozen in the flow."

IV. SAFETY FACTOR FOR THE TIME-DEPENDENT AXISYMMETRIC BAROTROPIC GAS FLOWS

In this paper, we investigate the safety factor q for the time-dependent axisymmetric barotropic gas flows. The term "safety factor" was first introduced in 1964 by Mercier in plasma physics[11] where q was connected with the stability (and hence safety) of axisymmetric plasma equilibria. In fluid dynamics, the term "pitch" $p = 2\pi q$ was first introduced for the stationary axisymmetric flows of ideal incompressible fluid with constant density ρ by Hicks[6,12] and was reintroduced later by Moffatt.[4]

Let us define the safety factor $q(\mathbf{x}, t)$ for the time-dependent axisymmetric vector fields $\mathbf{\Omega}(\mathbf{x}, t)$ (1.6) and prove its conservation along the barotropic compressible gas streaklines. Any closed integral curve of the vector field $\mathbf{\Omega}(r, z, t)$ lies on a torus $\mathbb{T}^2 = C \times \mathbb{S}^1$. Let the curve go m times a long way (along the circle $\mathbb{S}^1$) and n times a short way (along the closed curve C). The safety factor q for such a curve is defined in Ref. 11 as $q = m/n$. Other definitions of the pitch $p = 2\pi q$ for the fluid equilibria are given in Refs. 4 and 6.

We propose the following definition of the safety factor of the helical integral curves of the vector field $\mathbf{\Omega}(r, z, t)$ (1.6) on the time-dependent tori $T^2_{rw(t)} = C_{rw(t)} \times \mathbb{S}^1$, where t = const. Let $\tau(rw(t))$ be the period of the corresponding closed poloidal trajectory $C_{rw(t)}$ of system (2.1). The safety factor $q(r, z, t)$ of the helical integral curve is equal to the increment of the angle φ during one period $\tau(rw(t))$, divided by 2π,

$$q(r,z,t) = \frac{1}{2\pi}\oint_{C_{rw(t)}} \frac{d\varphi}{d\tau} d\tau = \frac{1}{2\pi}\int_0^{\tau(rw(t))} \frac{(u_z - v_r)[r(\tau), z(\tau), t]}{\rho(\tau) r(\tau)} d\tau. \tag{4.1}$$

Formula (4.1) defines the safety factor for both the infinite helical trajectories of system (2.1) and (2.2) on the torus $T^2_{rw(t)}$ and for the closed ones. The safety factor $q(r, z, t)$ has the same value for all integral curves of the vector field $\mathbf{\Omega}(r, z, t)$ on a given torus $T^2_{rw(t)}$ because the circle integral in (4.1) does not depend on the starting point. If the number $q(r, z, t)$ is rational $= m/n$, then during n periods $\tau(rw(t))$, the angle φ increases for $n \cdot 2\pi m/n = 2\pi m$. Therefore the corresponding curve on the torus $T^2_{rw(t)} = C_{rw(t)} \times \mathbb{S}^1$ is closed and makes m complete turns a long way and n turns a short way. Hence for the closed integral curves of the vector field $\mathbf{\Omega}(r, z, t)$, our definition (4.1) is equivalent to the one of Ref. 11. Such closed curves are called torus knots $K_{m,n}$.[22] The torus knots $K_{m,n} \subset T^2_{rw(t)}$ have a topological invariant that is the ratio m/n.[22]

V. SAFETY FACTOR $q(r, z, t)$ IS A MATERIAL CONSERVATION LAW FOR THE BAROTROPIC GAS FLOWS

Since the vector field $\mathbf{\Omega}(r, z, t)$ is frozen into the barotropic compressible gas flow, the same is true for the corresponding torus knots $K_{m,n}$. Hence during the dynamics of fluid, the main characteristic of the knots $m/n = q(r, z, t)$ is conserved because it is a topological invariant. Therefore all rational values of the safety factor $q(r, z, t)$ (4.1) are conserved along the fluid streaklines.

The continuity of function $q(r, z, t)$ (4.1) implies that tori $T^2_{rw(t)}$ with rational values of $q(r, z, t)$ are everywhere dense in each closed set $\mathcal{D}^3_k(t)$. Any irrational value of $q(r, z, t) = \xi$ is a limit of certain rational values $q_\ell(r, z, t) = m_\ell/n_\ell$:$\xi = q(r, z, t) = \lim_{\ell\to\infty} m_\ell/n_\ell$. Therefore the conservation during the barotropic gas dynamics of the rational values $q_\ell(r, z, t) = m_\ell/n_\ell$ and the continuity of function $q(r, z, t)$ (4.1) yield that all its irrational values $q(r, z, t) = \xi$ also are conserved. Hence the safety factor $q(r, z, t)$ and any continuous function of it, $F(q(r, z, t))$, are conserved along the compressible gas streaklines. Therefore the functions $F(q(r, z, t))$ form an infinite family of material conservation laws {for example, $\cos[nq(r, z, t)]$ and $q^n(r, z, t)$} which are defined inside the maximal vortex rings $\mathcal{R}^3_k(t)$ and blobs $\mathcal{S}^3_k(t)$ that are frozen into the fluid flow.

Remark 5. The safety factor $q(r, z, t)$ is constant on each torus $T^2_{rw(t)} = C_{rw(t)} \times \mathbb{S}^1$ because the circle integral in Eq. (4.1) does not depend on the starting point. The torus $T^2_{rw(t)} \subset \mathbb{R}^3$ is defined by the equation $rw(r, z, t) = rw = \text{const}$. Hence the function $q(r, z, t)$ (4.1) is constant on the compact level sets of function $rw(r, z, t)$. Nevertheless this does not automatically imply that the function $q(r, z, t)$ is a material conservation law. Indeed, for any differentiable functions $g(t)$ and $G(t, x)$, the differentiable functions $g(t)F(rw(r, z, t))$ and $G(t, rw(r, z, t))$ for any fixed t are constant on the level sets of function $rw(r, z, t)$. However the functions $g(t)F(rw(r, z, t))$ and $G(t, rw(r, z, t))$ evidently are not conserved along the fluid streaklines. Therefore the above presented proof is necessary. It implies that functions $q(r, z, t)$ and $rw(r, z, t)$ inside the vortex rings $\mathcal{R}^3_k(t)$ and vortex blobs $\mathcal{S}^3_\ell(t)$ are connected by some functional relations.

VI. NEW FUNCTIONAL INVARIANTS

The boundary $\partial D_k(t)$ is a 1-dimensional set $\mathcal{C}_k(t)$ that is invariant with respect to dynamical system (2.1) and hence satisfies the equation $rw(r, z, t) = \text{const}$ where the constant value does not depend on time because the function $rw(r, z, t)$ is a material conservation law. Thus we arrive at the new invariants

$$b_k = rw(r, z, t)\big|_{\mathcal{C}_k(t)}. \tag{6.1}$$

Invariants b_k are zeros at the boundaries of the vortex blobs $\mathcal{B}^3_\ell(t)$.

The function $rw(r, z, t)$ ($t = \text{const}$) of class C^2 has at least one point $c_k(t) = (r_k(t), z_k(t))$ of its local maximum or minimum in the interior of each maximal invariant compact set $D_k(t)$. The point $c_k(t)$ is an equilibrium point of system (2.1) and therefore is transported with the compressible gas flow as well as the corresponding vortex axis $\mathbb{S}_k(t) = c_k(t) \times \mathbb{S}^1$. Hence the values of the material conservation law $rw(r, z, t)$ at the points $c_k(t)$ are new functional invariants,

$$a_k = r_k(t)w(r_k(t), z_k(t), t). \tag{6.2}$$

Using the methods in Ref. 5, one can show that invariants b_k (6.1) and a_k (6.2) exist also for the inviscid compressible gas flows with arbitrary equation of state (not necessarily barotropic) and for the incompressible fluid flows with variable density $\rho(r, z, t)$.

Let $A_k(t)$ and $B_k(t)$ be the limits of the safety factor $q(r, z, t)$ at the point $c_k(t)$ and at the boundary $\mathcal{C}_k(t) = \partial D_k(t)$, respectively. Since the safety factor $q(r, z, t)$ is conserved along the barotropic gas streaklines, we get that its limits $A_k(t)$ and $B_k(t)$ are also conserved. Therefore the limits do not depend on time t and hence are the new invariants A_k and B_k of the axisymmetric barotropic gas flows (we call them the functional invariants). In Ref. 23, we have demonstrated by formulas (6.5) that the limits B_k can be $\pm\infty$.

Let us derive an exact formula for the invariants A_k. Suppose that for some constant time t the function $rw(r, z, t)$ has a non-degenerate local maximum or minimum at a point $c_k(t) = (r_k(t), z_k(t))$. This means that $(rw)_r(c_k(t)) = 0$, $(rw)_z(c_k(t)) = 0$, and the Hessian $\mathcal{H}(c_k(t)) = (rw)_{rr}(c_k(t))(rw)_{zz}(c_k(t)) - [(rw)_{rz}]^2(c_k(t)) > 0$. Then the neighboring curves $rw(r, z, t) = rw = \text{const}$ are closed curves $C_{rw(t)}$ encircling the point $c_k(t)$. The curves $C_{rw(t)}$ are closed trajectories of system (2.1). Let $\tau(rw(t))$ be their periods. In the limit $rw(r, z, t) \to rw(c_k(t))$, the closed curves $C_{rw(t)}$ tend to the equilibrium point $c_k(t)$:$(r = r_k(t), z = z_k(t))$. Applying formula (4.1), we find

$$A_k = \lim_{(r,z)\longrightarrow(r_k(t),z_k(t))} q(r, z, t) = \tau(rw(c_k(t)))\frac{(u_z - v_r)(c_k(t))}{2\pi\rho(c_k(t))r_k(t)}, \tag{6.3}$$

where $\tau(rw(c_k(t))) = \lim \tau(rw(t))$ when $rw(t) \to rw(c_k(t))$.

Dynamical system (2.1) near the equilibrium point $(r_k(t), z_k(t))$ is approximated by the system in variations[21]

$$\frac{d\delta r}{d\tau} = -a_{11}(t)\delta z - a_{12}(t)\delta r, \quad \frac{d\delta z}{d\tau} = a_{12}(t)\delta z + a_{22}(t)\delta r, \tag{6.4}$$

$$a_{11}(t) = h_k(rw)_{zz}(c_k(t)), \quad a_{12}(t) = h_k(rw)_{rz}(c_k(t)), \\ a_{22}(t) = h_k(rw)_{rr}(c_k(t)), \tag{6.5}$$

where $\delta r(\tau) = r(\tau) - r_k(t)$, $\delta z(\tau) = z(\tau) - z_k(t)$, and $h_k = [\rho(c_k(t))r_k(t)]^{-1}$. From formulas (6.5), we get

$$a_{11}(t)a_{22}(t) - a^2_{12}(t) = F_k(t) = h^2_k\mathcal{H}(c_k(t)) > 0. \tag{6.6}$$

Linear system (6.4) has quadratic first integral $Q(\delta r, \delta z) = a_{22}(t)(\delta r)^2 + 2a_{12}(t)\delta r\delta z + a_{11}(t)(\delta z)^2$ which in view of (6.6) is either positive or negative definite. Hence its level curves $Q(\delta r, \delta z) = \text{const}$ are nested ellipses and therefore all solutions to system (6.4) are periodic. Due to the scaling invariance of the linear system (6.4), all its solutions have the same period $\tau_k(t) = 2\pi/\sqrt{F_k(t)}$.[21]

From the general theory of dynamical systems,[21] it follows that the limit at $rw(r, z, t) \to rw(c_k(t))$ of the function of periods $\tau(rw(t))$ is the period $\tau_k(t)$ of the linear system in variations (6.4). Using formula (6.6), we find

$$\tau(rw_k(t)) = \lim_{rw(r,z,t)\longrightarrow rw(c_k(t))} \tau(rw(t)) = \frac{2\pi}{\sqrt{F_k(t)}} = \frac{2\pi\rho(c_k(t))r_k(t)}{\sqrt{\mathcal{H}(c_k(t))}}. \tag{6.7}$$

Using formulas (6.3) and (6.7) for the limit of the periods $\tau(rw(t))$ at $rw(r, z, t) \to r_k(t)w(c_k(t))$, we get

$$A_k = \frac{(u_z - v_r)(c_k(t))}{\sqrt{((rw)_{rr}(rw)_{zz} - [(rw)_{rz}]^2)(c_k(t))}}. \tag{6.8}$$

Formula (6.8) defines the explicit form of the new functional invariants of the axisymmetric barotropic gas flows.

New invariants b_k (6.1), a_k (6.2), A_k (6.8) evidently are functionally independent and do not depend on the gas density $\rho(\mathbf{x}, t)$.

Remark 6. The same invariants b_k (6.1), a_k (6.2), A_k (6.8) exist for the ideal incompressible fluid flows. For the latter, the incompressibility equation $\nabla \cdot \mathbf{V} = 0$ implies the existence of the flux function $\psi(r, z, t)$ and the relations $u = -r^{-1}\psi_z$, $v = r^{-1}\psi_r$ in Eq. (1.4). Substituting these formulas into Eq. (6.8), one gets the explicit form of invariants A_k for the ideal incompressible fluid flows with constant density ρ,

$$A_k = -\frac{\left(\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}\right)(c_k(t))}{r_k(t)\sqrt{((rw)_{rr}(rw)_{zz} - [(rw)_{rz}]^2)(c_k(t))}}. \tag{6.9}$$

Invariants b_k (6.1) and a_k (6.2) are the same.

VII. INFINITE FAMILIES OF CONSERVED QUANTITIES AND LOCAL CONSERVATION LAWS

From the fact that the safety factor $q(\mathbf{x}, t)$ for the barotropic gas flows and function $rw(\mathbf{x}, t)$ are material conservation laws, we get that for any differentiable function $G(x, y)$ of two variables x and y the composed function $G(q(\mathbf{x}, t), rw(\mathbf{x}, t))$ also is a material conservation law and hence satisfies the equation

$$\frac{\partial[G(q(\mathbf{x},t), rw(\mathbf{x},t))]}{\partial t} + \sum_{i=1}^{3} V_i(\mathbf{x},t)\frac{\partial[G(q(\mathbf{x},t), rw(\mathbf{x},t))]}{\partial x_i} = 0, \tag{7.1}$$

where $V_i(\mathbf{x}, t)$ are components of the compressible gas velocity.

The second equation (1.1) has the form

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \sum_{i=1}^{3}\frac{\partial[\rho(\mathbf{x},t)V_i(\mathbf{x},t)]}{\partial x_i} = 0. \tag{7.2}$$

Multiplying Eq. (7.1) with $\rho(\mathbf{x}, t)$, Eq. (7.2) with $G(q(\mathbf{x}, t), rw(\mathbf{x}, t))$, then adding the results, and applying the Leibnitz formula, we get an infinite family of local conservation laws

$$\frac{\partial[\rho(\mathbf{x},t)G(q(\mathbf{x},t), rw(\mathbf{x},t))]}{\partial t} + \sum_{i=1}^{3}\frac{\partial[\rho(\mathbf{x},t)V_i(\mathbf{x},t)G(q(\mathbf{x},t), rw(\mathbf{x},t))]}{\partial x_i} = 0. \tag{7.3}$$

For any differentiable function $G(x, y)$ of two variables x and y, the formulas

$$I_{kG} = \int_{\mathcal{D}_k^3(t)} \rho(\mathbf{x},t)G(q(\mathbf{x},t), rw(\mathbf{x},t))d\mathbf{x} \tag{7.4}$$

define an infinite family of conserved quantities for the axisymmetric Euler's equations (1.1). Indeed, the barotropic compressible gas flow preserves the composed material conservation laws $G(q(\mathbf{x}, t), rw(\mathbf{x}, t))$; it transforms the set $\mathcal{D}_k^3(t_1)$ into $\mathcal{D}_k^3(t)$ for $t > t_1$ and preserves the mass $\rho(\mathbf{x}, t)d\mathbf{x}$ of any infinitesimal domain transported with the gas flow. Hence the local conservation law (7.3) implies that the integral (7.4) is preserved by the gas dynamics. For example, let us consider Eq. (7.4) for $G(x, y) = \cos(nx)$. The functionals

$$I_{kn} = \int_{\mathcal{D}_k^3(t)} \rho(\mathbf{x},t)\cos(nq(\mathbf{x},t))d\mathbf{x} \tag{7.5}$$

are defined for any integers $n \geq 0$ and form an infinite family of conserved quantities for Eqs. (1.1). The conserved quantities $|I_{kn}|$ (7.5) for all $n \geq 0$ are bounded by the conserved masses (2.4) of gas in the domain $\mathcal{D}_k^3(t)$ because $|\cos(nq(\mathbf{x}, t))| \leq 1$. We get another infinite set of conserved quantities from Eq. (7.4) for $G(x, y) = y^m$,

$$J_{km} = \int_{\mathcal{D}_k^3(t)} \rho(\mathbf{x},t)[rw(\mathbf{x},t)]^m d\mathbf{x}. \tag{7.6}$$

In view of Eqs. (2.1) and (2.2), the helicity integral[4] over the domain $\mathcal{D}_k^3(t)$ has the form

$$H_k = \int_{\mathcal{D}_k^3(t)} \mathbf{V}\cdot(\nabla\times\mathbf{V})d\mathbf{x}. \tag{7.7}$$

Let us show that the integral invariants (7.5) are functionally independent from the helicity (7.7). Indeed, for any vortex ring or blob $\mathcal{D}_k^3(t)$, the integral invariants (7.5) as functions of the velocity $\mathbf{V}(r, z, t)$ (2.1) are mutually functionally independent because the integrals of the functions $\cos(ny)$ and $\cos(n_1 y)$ for $n \neq n_1$ are independent. Therefore if all integrals I_{kn} (7.5) with arbitrary $n \geq 1$ were functionally dependent on the helicity H_k (7.7), then I_{kn} would have mutually been functionally dependent on each other, but they are not. This proves that the integral invariants I_{kn} (7.5) for all $n \geq 1$ with possibly one exception are functionally independent from the helicity H_k (7.7).

VIII. AN EXTENSION OF KELVIN'S PROBLEM

The problem of classification of vortex knots for incompressible fluid flows with constant density ρ was first formulated and studied by Kelvin[24,25] and is now one of the classical problems of fluid mechanics.

The results of this paper imply that Kelvin's problem has an extension as a more general problem of classification of vortex knots for the barotropic flows of compressible gas. Indeed, as known,[2,3] the vector field $\mathbf{\Omega}(\mathbf{x}, t) = \text{curl}\,\mathbf{V}(\mathbf{x}, t)/\rho(\mathbf{x}, t)$ (1.2) is frozen into the barotropic gas flows. Therefore its closed trajectories (knots) are transported with the gas flow. Since vector fields $\mathbf{\Omega}(\mathbf{x}, t)$ and curl $\mathbf{V}(\mathbf{x}, t)$ are proportional, the closed trajectories of the vector field $\mathbf{\Omega}(\mathbf{x}, t)$ simultaneously are vortex knots. Hence the topological invariants of knots for the vector fields $\mathbf{\Omega}(\mathbf{x}, t)$ and curl $\mathbf{V}(\mathbf{x}, t)$ are invariants of the barotropic gas flows.

The moduli spaces of vortex knots were first introduced in our papers[27,28] for the steady flows of incompressible fluid with constant density ρ. They are also well defined for arbitrary time-dependent barotropic flows of compressible gas and ideal fluid. The moduli spaces in general are different for different time-dependent gas and fluid flows because the invariants b_k (6.1), a_k (6.2), and A_k [(6.8) and (6.9)] evidently depend on the particular flow.

The recent progress in the study of the classical Kelvin's problem is connected with the existence of many exact solutions to Euler equations (1.7) for the ideal incompressible fluid flows. The problem for the axisymmetric fluid and plasma equilibria was studied by Moffatt in Refs. 4, 13, and 15 where it was stated that for the axisymmetric fluid and plasma equilibria the limits of the pitch $p = 2\pi q$ at the vortex and magnetic axes are always infinite and the limits of $p = 2\pi q$ at the boundaries of the spheroids $\mathcal{S}_k^3$ are always zero. These two statements were incorrect and had led Moffatt to the erroneous conclusions[4,13,15] that all torus knots $K_{m,n}$ for all values of the parameter $q = m/n$ are realized as vortex and magnetic knots for the axisymmetric fluid and plasma equilibria. The correction was performed in Ref. 23; see also the Corrigendum.[26] We proved in Ref. 23 that by far not all torus knots are realized as vortex knots because the parameter $q = m/n$ of torus knots $K_{m,n}$ for different fluid equilibria takes values in certain subintervals of the axis $\mathbb{R}^1$. Therefore the moduli spaces of non-isomorphic vortex knots depend on concrete equilibria.

We presented in Refs. 27 and 28 the detailed studies of the moduli spaces of vortex knots for two different ideal fluid equilibria. The first of them[27] is the well-known spheromak Beltrami flow initially discovered and studied by Hicks[6] and later rediscovered as a plasma equilibrium by Chandrasekhar[29] and Woltjer[30] and studied by Moffatt in Refs. 4 and 15. We proved in Ref. 27 that for the spheromak flow in the first invariant spheroid the safety factor q takes values only in the interval $0.7152 < q < 0.8252$ and for the whole space $\mathbb{R}^3$ only in the interval $0.5 < q < 0.8252$. The sets of all rational points $m/n = q$ in these intervals form the corresponding moduli spaces of vortex knots. The second of the studied examples[28] belongs to a new family of the ideal incompressible fluid and plasma equilibria. The corresponding moduli spaces of vortex knots are the sets of all rational points in the intervals $0.4586 < q < 0.5847$ and $0.25 < q < 0.5847$, respectively, for the first invariant spheroid and for the whole space $\mathbb{R}^3$.

IX. CONCLUSION

The following main results were obtained in this article:

- We proved that the time-dependent safety factor $q(r, z, t)$ is a new material conservation law for the axisymmetric flows of barotropic compressible gas and ideal incompressible fluid with constant density ρ.
- We demonstrated the existence of the vortex rings $\mathcal{R}_k^3(t)$ and vortex blobs $\mathcal{S}_\ell^3(t)$ which are frozen into the axisymmetric gas and fluid flows.
- We constructed the infinite series of invariants

$$\text{Mass}\,\mathcal{R}_k^3(t),\ \text{Mass}\,\mathcal{B}_\ell^3(t),\ N_r,\ N_s,\ a_k,\ b_k,\ A_k,\ I_{kn},\ J_{km}, \tag{9.1}$$

  defined by Eqs. (2.4), (6.1), (6.2), (6.8), (6.9), (7.5), and (7.6) for the axisymmetric dynamics of the barotropic compressible gas and the ideal incompressible fluid. Here N_r and N_s are the total numbers of vortex rings $\mathcal{R}_k^3(t)$ and vortex blobs $\mathcal{S}_\ell^3(t)$, respectively. Invariants Mass$\mathcal{R}_k^3(t)$ and Mass$\mathcal{B}_\ell^3(t)$ are their masses (2.4).
- The existence of invariants (9.1) implies that the axisymmetric dynamics of barotropic compressible gas and the ideal incompressible fluid between two given states is possible only if the corresponding total numbers N_r of vortex rings are equal and their masses Mass$\mathcal{R}_k^3(t)$ are equal (the same for the total numbers N_s of vortex blobs and their masses), and all new invariants b_k (6.1), a_k (6.2), A_k (6.8), I_{kn} (7.5), and J_{km} (7.6) for them coincide. This evidently yields a stratification in the space of axisymmetric gas and fluid flows.

ACKNOWLEDGMENTS

The author thanks the referees for useful remarks.

REFERENCES

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).
[2] J. Serrin, "Mathematical principles of classical fluid mechanics," in *Fluid Dynamics* I, Encyclopedia of Physics, edited by C. Truesdell (Springer-Verlag, Berlin, Heidelberg, 1959), pp. 125–263.
[3] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer-Verlag, New York, 1979).
[4] H. K. Moffatt, "The degree of knottedness of tangled vortex lines," J. Fluid Mech. **35**(1), 117–129 (1969).
[5] O. Bogoyavlenskij, "Invariants of the axisymmetric flows of the inviscid gas and fluid with variable density," Z. Naturforsch. **73**(10), 931–937 (2018).
[6] W. M. Hicks, "Researches in vortex motion. Part III. On spiral or gyrostatic vortex aggregates," Philos. Trans. R. Soc., A **192**, 33–99 (1899).
[7] O. Kelbin, A. F. Cheviakov, and M. Oberlack, "New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows," J. Fluid Mech. **721**, 340–366 (2013).
[8] O. Bentata, D. Anne-Archard, and P. Brancher, "Experimental study of low inertia vortex rings in shear-thinning fluids," Phys. Fluids **30**, 113103 (2018).
[9] D. Barz, M. Scholz, and S. Hardt, "Electrokinetic manipulation of the von Karman vortex street in the wake of a confined cylinder. I. DC electric field," Phys. Fluids **30**, 082004 (2018).
[10] R. H. Hernandez and T. Reyes, "Symmetrical collision of multiple vortex rings," Phys. Fluids **29**, 103604 (2017).
[11] C. Mercier, "Equilibre et satabilite d'un systeme toroidal magnetohydrodynamique au voisinage d'un axe magnetique," Fusion Nucl. **4**, 213–226 (1964).

[12]W. M. Hicks, "Researches in vortex motion. Part III. On spiral or gyrostatic vortex aggregates," Proc. R. Soc. London **62**, 332–338 (1898).
[13]H. K. Moffatt, "Generalised vortex rings with and without swirl," in *Vortex Motion. Proceedings of the IUTAM Symposium on Fundamental Aspects of Vortex Motion, Tokyo, Japan, 31 August–4 September* 1987 (North-Holland Publishing House, Amsterdam, 1988), pp. 22–30.
[14]D. Biskamp, *Nonlinear Magnetohydrodynamics* (Cambridge University Press, Cambridge, 1997).
[15]H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978).
[16]J. R. Cary and R. G. Littlejohn, "Noncanonical Hamiltonian mechanics and its applications to magnetic field line flow," Ann. Phys. **151**, 1–34 (1983).
[17]M. A. Berger, "Introduction to magnetic helicity," Plasma Phys. Controlled Fusion **41**, B167–B175 (1999).
[18]P. M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, 2008).
[19]P. M. Bellan, *Spheromaks: A Practical Application of Magnetohydrodynamic Dynamos and Plasma Self-Organization* (Imperial College Press, 2000).
[20]O. Bogoyavlenskij, "Invariants of the axisymmetric plasma flows," Z. Naturforsch. **73**(6a), 539–546 (2018).
[21]E. L. Ince, *Ordinary Differential Equations* (Dover Publications, New York, 1956).
[22]R. H. Crowell and R. H. Fox, *Introduction to Knot Theory* (Ginn and Company, Boston, 1963).
[23]O. Bogoyavlenskij, "Counterexamples to Moffatt's statements on vortex knots," Phys. Rev. E **95**, 043104 (2017), 14p.
[24]W. Thomson (Lord Kelvin), "On vortex motion," Trans. R. Soc. Edinburgh **25**, 217–260 (1869).
[25]W. Thomson (Lord Kelvin), "On vortex statics," Proc. R. Soc. Edinburgh **14**, 115–125 (1875).
[26]H. K. Moffatt, "The degree of knottedness of tangled vortex lines–Corrigendum," J. Fluid Mech. **830**, 821–822 (2017).
[27]O. Bogoyavlenskij, "Vortex knots for the spheromak fluid flow and their moduli spaces," J. Math. Anal. Appl. **450**, 21–47 (2017).
[28]O. Bogoyavlenskij, "Moduli spaces of vortex knots for an exact fluid flow," J. Math. Phys. **58**, 013101-01–013101-28 (2017).
[29]S. Chandrasekhar, "On force-free magnetic fields," Proc. Natl. Acad. Sci. U. S. A. **42**, 1–5 (1956).
[30]L. Woltjer, "The Crab Nebula," Bull. Astron. Inst. Neth. **14**, 39–80 (1958).
[31]In 1969, Moffatt wrote on p. 120 of Ref. 4: "*Under the barotropic condition* $p = p(\rho)$. . ., *the vorticity equation takes the well-known form* $D/Dt(\omega/\rho) = (\omega/\rho)\cdot\nabla\mathbf{u}$ (13)." The equivalent form of this equation for the "*isentropic*" gas flows was published in 1979 in Ref. 3.
[32]Dynamics of vortex rings in a viscous incompressible fluid was studied by experiments in Ref. 8 and numerically in Ref. 9. Reference 10 is devoted to the numerical investigation of the collisions of symmetrically positioned vortex rings in a viscous fluid. In the present paper, we study the barotropic axisymmetric flows of compressible gas and flows of ideal incompressible fluid at the assumption that the viscosity effects can be neglected.