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Vortex knots for the spheromak fluid flow and their moduli spaces

Oleg Bogoyavlenskij

Queen's University, Kingston, Canada

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ABSTRACT

New exact solutions to the Euler hydrodynamics equations are constructed. A method for the study of vortex knots is developed for a special class of ideal fluid flows – the axisymmetric ones satisfying the Beltrami equation $\operatorname{curl}\mathbf{V}(\mathbf{x}) = \lambda\mathbf{V}(\mathbf{x})$. The method is based on a construction of the moduli spaces of vortex knots $\mathcal{S}(\mathbb{R}^3)$. Applying the method to the spheromak fluid flow we demonstrate that only those torus knots $K_{p,q}$ are realized as vortex knots for which p/q belongs to the interval $I_1: 0.5 < \tau < M_1 \approx 0.8252$. We prove that each torus knot $K_{p,q}$ with $1/2 < p/q < 1/\sqrt{2}$ is realized on countably many invariant tori $\mathbb{T}^2 \subset \mathbb{R}^3$, while torus knots with $1/\sqrt{2} < p/q < M_1$ are realized only on finitely many tori. The moduli spaces of vortex knots $\mathcal{S}_m(\mathbb{B}_a^3)$ $(m = 1, 2, \cdots)$ are constructed for the spheromak fluid flows inside a ball $\mathbb{B}_a^3$ of radius a.

1. Introduction

1.1. We study exact solutions to the Euler equations of dynamics of an ideal incompressible fluid

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\frac{1}{\rho}\operatorname{grad} p, \qquad \operatorname{div}\mathbf{V} = 0 \tag{1.1}$$

and the corresponding vortex knots. Here $\mathbf{V}(t, \mathbf{x})$ is the fluid velocity vector field, $p(t, \mathbf{x})$ the pressure, the density ρ is supposed to be constant.

As known, the Euler equations yield the Helmholtz equation [4,35] for the vorticity:

$$\frac{\partial \operatorname{curl}\mathbf{V}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \operatorname{curl}\mathbf{V}), \tag{1.2}$$

which implies that the vorticity vector field $\operatorname{curl}\mathbf{V}$ "is frozen in the flow" (or is transformed in time by the flow diffeomorphisms). This yields that any knot formed by a closed vortex line at a time t is transformed by the flow into an isotopic knot.

E-mail address: bogoyavl@mast.queensu.ca.

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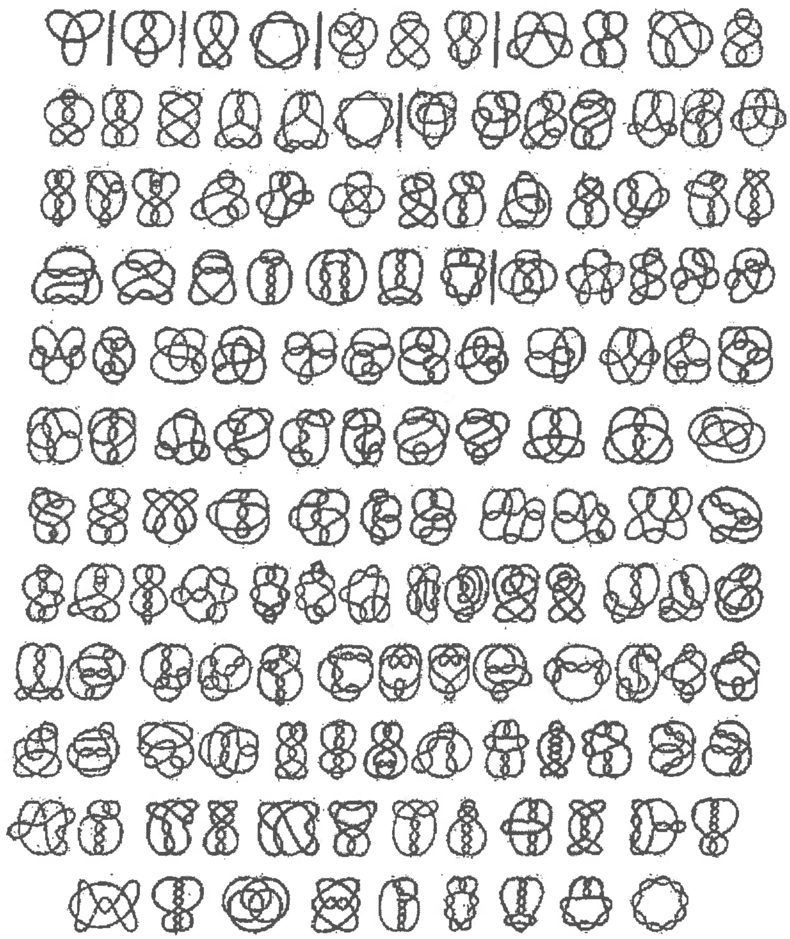

Fig. 1. Figures of knots obtained from [30].

We study a well-known problem that was around since the works by Kelvin [31–33]:
For a concrete solution to the Euler equations (1.1), to classify all its vortex knots $\mathcal{K} \subset \mathbb{R}^3$, up to the isotopy equivalence.

Definition 1. Moduli space $\mathcal{S}(\mathcal{D})$ of vortex knots for a solution to equations (1.1) in an invariant domain $\mathcal{D} \subset \mathbb{R}^3$ is a set that is in a one-to-one correspondence with all classes of isotopy equivalence of vortex knots $\mathcal{K} \subset \mathcal{D}$ existing for the considered solution at a given time t.

Applying Helmholtz theorem [35] we see that the moduli space $\mathcal{S}(\mathcal{D})$ does not depend on time t and hence is an invariant of the ideal fluid flow. The moduli space $\mathcal{S}(\mathcal{D})$ evidently does exist for any hydrodynamic flow and always is either finite or countable. Indeed, this follows from the fact that there is only a countable set of isotopy classes of smooth knots in $\mathbb{R}^3$ [11,27].

Remark 1. According to the historical studies by Epple [13], the works by Helmholtz [35], Kelvin [31–33] and Tait [29,30] on vortex knots published in 1850s–1880s had laid the foundation of the topological methods of hydrodynamics. Fig. 1 illustrates the Tait's classification [30] of simplest knots.

1.2. Equations of viscous magnetohydrodynamics in case of constant density ρ have the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\frac{1}{\rho} \operatorname{grad} \tilde{p} + \frac{1}{\rho\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B} + \nu \Delta \mathbf{V}, \tag{1.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}), \quad \operatorname{div} \mathbf{V} = 0, \quad \operatorname{div} \mathbf{B} = 0,$$

where $\mathbf{B}$ is the magnetic field, μ the magnetic permeability, ν the kinematic viscosity and Δ the Laplace operator. As shown by Newcomb [25], the second equation (1.3) implies that the magnetic field $\mathbf{B}$ is transformed in time by the flow diffeomorphisms (or "is frozen in the flow").

Definition 2. Moduli space $\mathcal{S}(\mathcal{D})$ of magnetic field $\mathbf{B}$ knots for a solution to equations (1.3) in an invariant with respect to the vector fields $\mathbf{B}$ and $\mathbf{V}$ domain $\mathcal{D} \subset \mathbb{R}^3$ is a set that is in a one-to-one correspondence with all classes of isotopy equivalence of knots $\mathcal{K} \subset \mathcal{D}$ formed by the closed magnetic field $\mathbf{B}$ lines for the considered solution at a given time t.

The frozenness of the magnetic field $\mathbf{B}$ lines in the magnetohydrodynamics flow yields that the moduli space $\mathcal{S}(\mathcal{D})$ does not depend on time t and hence is an invariant of the solution to equations (1.3).

Thus we arrive at the two problems of finding the moduli spaces $\mathcal{S}(\mathcal{D})$ of *(a)* vortex knots for solutions to the hydrodynamics equations (1.1) and *(b)* knots formed by the closed magnetic field $\mathbf{B}$ lines for solutions to the MHD equations (1.3).

Remark 2. The authors of [12] published an existence Theorem 1.1 which states that for any link $L \subset \mathbb{R}^3$ and any $\lambda \neq 0$ *"one can transform L by a C^1 diffeomorphism Φ of $\mathbb{R}^3$ arbitrarily close to the identity in any C^r norm, so that $\Phi(L)$ is a set of stream lines of a Beltrami field u, which satisfies* $\operatorname{curl} u = \lambda u$ *in $\mathbb{R}^3$"*. It is evident that the existence Theorem 1.1 [12] does not give solution to the problem of finding of the moduli space $\mathcal{S}(\mathbb{R}^3)$ of *all* non-isotopic vortex knots for a *concrete* exact fluid flow. This problem was not discussed in paper [12] and it is not the inverse problem for the one studied in [12].

1.3. We consider the steady Euler equations in the Bernoulli form

$$\mathbf{V} \times \operatorname{curl} \mathbf{V} = \operatorname{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2\right), \quad \operatorname{div} \mathbf{V} = 0 \tag{1.4}$$

and the plasma equilibrium equations in magnetohydrodynamics ($\mathbf{V} = 0$):

$$\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad}(-\mu\tilde{p}), \quad \operatorname{div} \mathbf{B} = 0. \tag{1.5}$$

Due to the evident equivalence of equations (1.4) and (1.5) any result concerning one of them is equally applicable to another.

Kruskal and Kulsrud in their pioneering paper [18] of 1958 proved for equations (1.5) that surfaces $\tilde{p}(\mathbf{x}) = \text{const.}$ *"by $\mathbf{B} \cdot \nabla p = 0$ are "magnetic surfaces", in the sense that they are made up of lines of magnetic force, and simultaneously by $\mathbf{j} \cdot \nabla p = 0$ they are "current surfaces". If such a surface lies in a bounded volume of space and has no edges and either $\mathbf{B}$ or $\mathbf{j}$ nowhere vanishes on it then by a well-known theorem [1] it must be a toroid (by which we mean a topological torus) or a Klein bottle. The latter, however is not realizable in physical space."*

In the 1959 paper [26] Newcomb stated *"It is easy to verify that the lines of force on a pressure surface are closed if and only if $\mathbf{i}(P)/2\pi$ is rational; if it is irrational, the lines of force cover the surface ergodically."* Here $\mathbf{i}(P)$ is the rotational transform connected with the safety factor $\mathbf{q}(P)$ [20] by the relation $\mathbf{q}(P) = 2\pi/\mathbf{i}(P)$. The safety factor $\mathbf{q}$ is binded with stability of the considered plasma equilibrium [14,20].

In 1965 the analogous results for the equivalent equations (1.4) were published by Arnold in [2] and in [3] where he added to [18,26] a statement that if a Bernoulli's surface M intersects the boundary of the invariant domain D then M has *"co-ordinates of the ring"* and *"all streamlines on M are closed."*

Remark 3. The steady equations in ideal magnetohydrodynamics (with vanishing viscosity ν) have the form:

$$\mathbf{V}\times\operatorname{curl}\mathbf{V}-\frac{1}{\rho\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B}=\operatorname{grad}\left(\frac{\tilde{p}}{\rho}+\frac{1}{2}|\mathbf{V}|^2\right), \tag{1.6}$$
$$\operatorname{curl}(\mathbf{V}\times\mathbf{B})=0,\quad \operatorname{div}\mathbf{V}=0,\quad \operatorname{div}\mathbf{B}=0.$$

For the collinear vector fields $\mathbf{V}(\mathbf{x})=\gamma\mathbf{B}(\mathbf{x})$, equations (1.6) reduce to

$$(\gamma^2\rho\mu-1)\mathbf{B}\times\operatorname{curl}\mathbf{B}=\operatorname{grad}\left(\mu\tilde{p}+\frac{\gamma^2\rho\mu}{2}|\mathbf{B}|^2\right),\quad \operatorname{div}\mathbf{B}=0. \tag{1.7}$$

It is evident that for any constant $\gamma\neq\pm1/\sqrt{\rho\mu}$ equations (1.7) are equivalent to equations (1.4) and (1.5).

1.4. We study in this paper the special steady solutions which do not satisfy the Kruskal and Kulsrud conditions [18] because for them pressure $\tilde{p}$ is identically constant. We consider vector fields $\mathbf{B}(\mathbf{x})$ satisfying the Beltrami equation

$$\operatorname{curl}\mathbf{B}=\lambda\mathbf{B}. \tag{1.8}$$

Beltrami vector fields (1.8) evidently are solutions to equation (1.5) with $\tilde{p}\equiv$ const. [9,10,37]. The equivalent hydrodynamics equilibria (1.4) are defined by equations

$$\operatorname{curl}\mathbf{V}=\lambda\mathbf{V},\quad p=C-\frac{\rho}{2}|\mathbf{V}|^2. \tag{1.9}$$

Chandrasekhar [9] and Chandrasekhar and Kendall [10] and Woltjer [37] presented an infinite basis of solutions to the Beltrami equation (1.8) in terms of the Bessel and Legendre functions. Results of [9,10,37] were used in many publications, see for example [8,19,24,28].

In papers [6,7] we derived an integral representation of Beltrami fields (1.8) which depends on an arbitrary vector field $\mathbf{T}(\mathbf{x})$ tangent to the unit sphere $\mathbb{S}^2$.

The spectrum and the eigenvector fields for the boundary eigenvalue problems for the operator curl on different domains in $\mathbb{R}^3$ were studied in works [8,19,24,34,38], that use the Chandrasekhar, Kendall [10] and Woltjer [37] representation of Beltrami fields in terms of Bessel and Legendre functions.

1.5. In hydrodynamics the vortex field $\operatorname{curl}\mathbf{V}(\mathbf{x})$ is frozen in the flow; for the MHD solutions the magnetic field $\mathbf{B}(\mathbf{x})$ is frozen in the flow. For Beltrami fields (1.8), (1.9) the vortex field and magnetic field are proportional. Therefore our study is equally applicable to both equations (1.8) and (1.9). For concreteness, we consider the hydrodynamic flow $\mathbf{V}(\mathbf{x})$.

In Sections 2–3 of this paper we construct exact solutions for the axisymmetric fluid flows with velocity vector fields

$$\mathbf{V}=-\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_r+\frac{1}{r}\frac{\partial\psi}{\partial r}\hat{\mathbf{e}}_z+\frac{\lambda\psi}{r}\hat{\mathbf{e}}_\varphi \tag{1.10}$$

in the cylindrical coordinates r, z, φ. The solutions are presented in terms of elementary functions and not in terms of the Bessel and Legendre functions as in [9,10,37]. Here $\psi(r,z)$ is the Stokes stream function [4] and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\varphi$ are the unit orts in the directions of coordinates r, z, φ.

Remark 4. The *spheromak* plasma equilibrium $\mathbf{B}_s(\mathbf{x})$ was derived for the first time by Woltjer [37] and was applied by him to model the Crab Nebula. The corresponding magnetic field has form (1.10) with the flux function

$$\psi(r,z) = Ar^2 \left(\frac{a}{R}\right)^{3/2} J_{3/2}(\lambda R), \tag{1.11}$$

where $R = \sqrt{r^2+z^2}$ and $J_{3/2}(u)$ is the Bessel function of order 3/2. The spheromak magnetic field $\mathbf{B}_s(\mathbf{x})$ (1.10)–(1.11) is considered either in the whole space $\mathbb{R}^3$ or in a ball $R \le a$ provided that parameter λ is defined by the equation $J_{3/2}(\lambda a) = 0$. The term *"spheromak"* for the plasma equilibrium inside a sphere was first introduced by Rosenbluth and Bussac in 1979 paper [28]. The review of the studies of the spheromak-like plasma configurations is given in [16].

1.6. In Section 4 we present a method for constructing the moduli spaces of vortex knots $\mathcal{S}(\mathcal{D})$ for the axisymmetric Beltrami fields (1.10). The method is based on the investigation of the functions of periods $\tau_k(H)$ *(in a special time variable* τ*)* of closed trajectories of certain 2-dimensional dynamical system in invariant domains $\mathcal{D}_k \subset \mathbb{R}^2$. The system is obtained by: *(1)* a reduction of the main axisymmetric dynamical system in $\mathbb{R}^3$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \operatorname{curl} \mathbf{V}(\mathbf{x}) \tag{1.12}$$

for the considered fluid flow in the cylindrical coordinates r, z, φ to a 2-dimensional system in the plane $\mathbb{R}^2$ with coordinates r, z; *(2)* a special choice of the time variable τ satisfying equation $\mathrm{d}\tau/\mathrm{d}t = H(r,z)/(2\pi r^2)$ that becomes singular at the boundaries $H(r,z) = 0$ of the invariant domains $\mathcal{D}_k$. The function $H(r,z)$ is a first integral of the dynamical system (1.12) and coincides with the stream function $\psi(r,z)$ (1.10). The function of periods $\tau(H)$ is connected with the safety factor $\mathbf{q}(H)$ and the pitch $p(H)$ of the corresponding helical trajectories of $\operatorname{curl} \mathbf{V}(\mathbf{x})$ on the invariant tori $\mathbb{T}^2_H$ by the relations

$$\tau(H) = \mathbf{q}(H) = \frac{1}{2\pi} p(H). \tag{1.13}$$

All vortex knots for the derived exact axisymmetric fluid flows are torus knots $K_{p,q}$ that correspond to the rational values of function of periods $\tau(H) = p/q$. Closed vortex lines with $\tau(H) = n$ and $\tau(H) = 1/n$ where n is any integer form "unknots" and are isotopic to a trivial embedding of a circle. Therefore the moduli space $\mathcal{S}(\mathcal{D})$ of vortex knots is the set of all rational numbers in the range of function $\tau(H)$ where numbers n and $1/n$ are excluded.[1]

Note that one cannot find explicitly even a single value of any function of periods $\tau_k(H)$. However we have calculated using two limiting procedures the exact values of the lower and upper bounds for the ranges of the continuous functions $\tau_k(H)$ for each invariant domain $\mathcal{D}_k$. The both bounds are finite positive numbers and are presented by exact formulae containing the roots of equations $\tan r = Q(r)$ where $Q(r)$ are certain rational functions. Our construction of the moduli spaces $\mathcal{S}(\mathcal{D})$ is based on the derived lower and upper bounds for all functions of periods $\tau_k(H)$ for $k = 1, 2, \cdots$.

The methods of this paper can be applied to some other partial differential equations studied in [5].

In Sections 5–7 the ranges and domains of functions of periods $\tau_k(H)$ in the time variable τ are obtained in exact form for the spheromak solution.

In Section 8 we demonstrate that the moduli space of vortex knots $\mathcal{S}(\mathbb{R}^3)$ for the steady spheromak solution $\mathbf{V}(\mathbf{x})$ to the Euler equations (1.1) in $\mathbb{R}^3$ is naturally isomorphic to the set of all rational numbers p/q in the interval

$$I_1: \quad 0.5 < \tau < M_1 = \frac{r_{1*}}{\sqrt{2(r_{1*}^2 - 2)}} \approx 0.8252, \tag{1.14}$$

[1] For the analogous MHD solutions the moduli space $\mathcal{S}(\mathcal{D})$ of magnetic field knots is the set of all rational numbers in the range of the safety factor $\mathbf{q}(H)$ where numbers n and $1/n$ are excluded.

where $r_{1*} \approx 2.7437$ is the first positive solution to equation $\tan r = r/(1-r^2)$. The interval I_1 (1.14) is the complete range of function of periods $\tau(H)$ in the whole space $\mathbb{R}^3$. The complete range of the pitch function $p(H)$ is $2\pi \cdot I_1$.

The same moduli space $\mathcal{S}(\mathbb{R}^3)$ exists for the magnetic field $\mathbf{B}$ knots for the spheromak plasma equilibrium solution to equations (1.5) and for the MHD equilibrium solutions to equations (1.7).

1.7. Moffatt stated in [21] on page 129 that for the spheromak fluid flow $\mathbf{V}_s$ (that is one of the considered flows) in the first invariant ball $\mathbb{B}^3_{a_1}$ the pitch p (that is $p = 2\pi\mathbf{q}$) takes all values from zero to infinity, hence $\mathbf{q} \in (0, \infty)$. The same claim is made in [22] (see pp. 30–31) for the spheromak magnetic field $\mathbf{B}_s$. As a consequence Moffatt affirmed that for the spheromak magnetic field $\mathbf{B}_s$ (and for the analogous fluid flow $\mathbf{V}_s$) the closed magnetic field lines (vortex lines) represent all torus knots $K_{p,q}$ for any rational values of $\mathbf{q} = p/q$. The analogous statements are made on page 29 of [23] for the general axisymmetric fields $\mathbf{B}(r, z)$. We prove in Section 7 that these statements of [21–23] are incorrect and that the safety factor $\mathbf{q}$ for the spheromak field in $\mathbb{B}^3_{a_1}$ takes values only in an interval of a small length $\ell \approx 0.110017$. Therefore the Moffatt statements of [21–23] that the pitch p (or the safety factor $\mathbf{q} = p/2\pi$) takes all values from the infinite interval $(0, \infty)$ do not correspond to the facts.

For the spheromak flow $\mathbf{V}_s(\mathbf{x})$ (1.10)–(1.11) in the whole Euclidean space $\mathbb{R}^3$, equation (1.14) yields that only those torus knots $K_{p,q}$ are realized as vortex knots for which $0.5 < p/q < 0.8252$.

1.8. In Section 9 we construct the moduli spaces of vortex knots $\mathcal{S}_m(\mathbb{B}^3_a)$ $(m = 1, 2, \cdots)$ for the spheromak fluid flows $\mathbf{V}_{1m}(\mathbf{x})$ inside a ball $\mathbb{B}^3_a$ of radius a which are tangent to the boundary sphere $\mathbb{S}^2_a$. The integer $m \geq 1$ is equal to the number of invariant under the flow $\mathbf{V}_{1m}(\mathbf{x})$ spheres $\mathbb{S}^2_{a_k}$ of radii $a_k \leq a$ inside the ball $\mathbb{B}^3_a$. We show that the moduli spaces of vortex knots $\mathcal{S}_m(\mathbb{B}^3_a)$ and $\mathcal{S}_\ell(\mathbb{B}^3_a)$ are different for $m \neq \ell$, do not depend on the radius a and that $\mathcal{S}_m(\mathbb{B}^3_a)$ approximate $\mathcal{S}(\mathbb{R}^3)$ when $m \longrightarrow \infty$.

2. Exact solutions to the Euler equations

2.1. The change of variables $\tilde{x}_i = \lambda x_i$ yields the equivalence of eigenvector fields for the curl operator (1.9) in $\mathbb{R}^3$ with eigenvector fields corresponding to the eigenvalue $\lambda = 1$. Therefore we will consider first Beltrami fields with $\lambda = 1$:

$$\operatorname{curl} \mathbf{V}(\mathbf{x}) = \mathbf{V}(\mathbf{x}). \tag{2.1}$$

We will return to the case of general eigenvalues λ in Section 9.

As is known [9,10], for any constant vector $\mathbf{A}$ the vector fields

$$\mathbf{V}(\mathbf{x}) = \operatorname{curl}(f(\mathbf{x})\mathbf{A}) + \operatorname{curl}\operatorname{curl}(f(\mathbf{x})\mathbf{A}), \quad \Delta f(\mathbf{x}) + f(\mathbf{x}) = 0 \tag{2.2}$$

satisfy Beltrami equation (2.1). Here Δ is the Laplace operator and $f(\mathbf{x})$ is its eigenfunction.

For the spherically symmetric eigenfunctions $f(\mathbf{x}) = f(R)$ the Helmholts' equation $\Delta f = -f$ takes the form

$$f'' + \frac{2}{R} f' = -f. \tag{2.3}$$

The analytic spherically symmetric eigenfunctions $f(R)$ are proportional to

$$G_1(R) = \frac{\sin R}{R}. \tag{2.4}$$

Using identity $\operatorname{curl}\operatorname{curl}\mathbf{S}(\mathbf{x}) = -\Delta\mathbf{S}(\mathbf{x}) + \operatorname{grad}\operatorname{div}\mathbf{S}(\mathbf{x})$ and equation (2.3) we find for vector field (2.2):

$$\mathbf{V}(\mathbf{x}) = \operatorname{curl}(f\mathbf{A}) + f\mathbf{A} + \operatorname{grad}\operatorname{div}(f\mathbf{A}). \tag{2.5}$$

Using identities $\operatorname{curl}(f\mathbf{A}) = \operatorname{grad} f \times \mathbf{A}$ and $\operatorname{div}(f\mathbf{A}) = \operatorname{grad} f \cdot \mathbf{A} = \nabla_{\mathbf{A}} f$, we transform vector field $\mathbf{V}(\mathbf{x})$ (2.2)–(2.5) into

$$\mathbf{V}(\mathbf{x}) = f\mathbf{A} + (\operatorname{grad} f) \times \mathbf{A} + \operatorname{grad}(\nabla_{\mathbf{A}} f). \tag{2.6}$$

Derivative of $f(R) = G_1(R)$ (2.4) in the direction of vector $\mathbf{A}$ is $\nabla_{\mathbf{A}} G_1 = \operatorname{grad} G_1 \cdot \mathbf{A} = (\mathbf{x}\cdot\mathbf{A})G_2$, where

$$G_2 = G_2(R) = \frac{1}{R}\frac{\mathrm{d}G_1}{\mathrm{d}R} = \frac{1}{R^2}\left(\cos R - \frac{\sin R}{R}\right). \tag{2.7}$$

Function $G_2(R)$ is analytic. Indeed, using classical series for $\sin R$ and formulae (2.4), (2.7) we get the convergent series

$$G_2(R) = -2\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)}{(2n+3)!}R^{2n}. \tag{2.8}$$

Hence we find $G_2(0) = -1/3$. Using Watson's formula for the Bessel functions $J_{n+1/2}(R)$ [36], p. 56, we find $G_2(R) = -\sqrt{\pi/2}R^{-3/2}J_{3/2}(R)$.

Lemma 1. *Function $G_2(R)$ (2.7) satisfies equation*

$$G_2'' + \frac{4}{R}G_2' = -G_2. \tag{2.9}$$

Proof. Function $G_1(R)$ (2.4) satisfies equation (2.3). Substituting $G_1' = RG_2$ (2.7) into equation (2.3) we get $(RG_2)' + 2G_2 = -G_1$ that is $RG_2' + 3G_2 = -G_1$. Differentiating this equation and substituting $G_1' = RG_2$ (2.7) we get $RG_2'' + 4G_2' = -RG_2$. This equation evidently yields equation (2.9). □

2.2. For function $f(\mathbf{x}) = G_1(R)$, using $\operatorname{grad} G_1(R) = R^{-1}G_1'(R)\mathbf{x}$ we get the vector field $\mathbf{V}_1(\mathbf{x},\mathbf{A})$ (2.6):

$$\mathbf{V}_1(\mathbf{x},\mathbf{A}) = G_1\mathbf{A} + \frac{G_1'}{R}\mathbf{x}\times\mathbf{A} + \frac{1}{R}\left(\frac{G_1'}{R}\right)'(\mathbf{x}\cdot\mathbf{A})\mathbf{x} + \frac{G_1'}{R}\mathbf{A}. \tag{2.10}$$

Using equation (2.3) for function $f(R) = G_1(R)$ we find

$$\frac{1}{R}\frac{\mathrm{d}G_2}{\mathrm{d}R} = \frac{1}{R}\left(\frac{G_1'}{R}\right)' = -\frac{1}{R^2}\left(3\frac{G_1'}{R} + G_1\right) = -\frac{1}{R^2}(3G_2 + G_1) = G_3. \tag{2.11}$$

Function $G_3(R)$ is analytic. Indeed, using equation (2.11): $G_3 = G_2'/R$ and series (2.8) we derive the convergent series $G_3(R) = 4\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)}{(2n+5)!}R^{2n}$. Hence we get $G_3(0) = 1/15$.

By induction we define functions

$$G_N(R) = \frac{1}{R}\frac{\mathrm{d}G_{N-1}(R)}{\mathrm{d}R}, \qquad N = 2, 3, \cdots. \tag{2.12}$$

Formulae (2.8), (2.12) imply that all functions $G_N(R)$ are analytic and $G_N(0) \neq 0$. For functions $G_N(R)$ we evidently have

$$\frac{\partial G_N}{\partial z} = zG_{N+1}, \tag{2.13}$$

in the Cartesian coordinates x, y, z.

Substituting formulae (2.7) and (2.11) into (2.10) we get for vector $\mathbf{A} = \hat{\mathbf{e}}_z$

$$\mathbf{V}_1(\mathbf{x}) = (G_1 + G_2)\,\hat{\mathbf{e}}_z + G_2\mathbf{x}\times\hat{\mathbf{e}}_z + (\mathbf{x}\cdot\hat{\mathbf{e}}_z)G_3\mathbf{x} = \tag{2.14}$$
$$(yG_2 + xzG_3)\hat{\mathbf{e}}_x + (-xG_2 + yzG_3)\hat{\mathbf{e}}_y + (G_1 + G_2 + z^2G_3)\hat{\mathbf{e}}_z,$$

where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$ are the Cartesian unit orts. This formula together with the pressure $\tilde{p} =$ const. gives an exact solution to the plasma equilibrium equations (1.5). Vector field (2.14) coincides with the *"spheromak"* magnetic field $\mathbf{B}_s(\mathbf{x})$ (1.10)–(1.11) which was first derived by a different method by Woltjer in [37]. The same vector field $\mathbf{V}_1(\mathbf{x})$ (2.14) together with the pressure $p(\mathbf{x}) = C - \rho|\mathbf{V}_1(\mathbf{x})|^2/2$ defines the spheromak exact solutions $\mathbf{V}_s(\mathbf{x})$ to the steady Euler equations (1.4).

2.3. Let us use cylindrical coordinates r, φ, z which are connected with the Cartesian coordinates x, y, z by relations $r = \sqrt{x^2+y^2}$, $x = r\cos\varphi$, $y = r\sin\varphi$. The corresponding orthogonal unit orts are

$$\hat{\mathbf{e}}_z, \quad \hat{\mathbf{e}}_r = \cos\varphi\hat{\mathbf{e}}_x + \sin\varphi\hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_\varphi = -\sin\varphi\hat{\mathbf{e}}_x + \cos\varphi\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z\times\hat{\mathbf{e}}_r. \tag{2.15}$$

The eigenfunction equation for Laplace operator for the axisymmetric functions $f(r,z)$ takes the form

$$\Delta f(r,z) = f_{rr} + \frac{1}{r}f_r + f_{zz} = -f, \tag{2.16}$$

where f_r, f_{rr}, f_{zz} mean partial derivatives. The change of variables $\tilde{r} = r/\lambda$, $\tilde{z} = z/\lambda$ establishes equivalence of eigenfunctions (2.16) with eigenfunctions corresponding to an arbitrary eigenvalue $-\lambda^2$. Therefore it is sufficient to consider only eigenfunctions (2.16) with eigenvalue -1.

The eigenfunction $G_1(R)$ (2.4) has the form

$$G_1(r,z) = \frac{\sin\sqrt{r^2+z^2}}{\sqrt{r^2+z^2}}. \tag{2.17}$$

Since equation (2.16) is invariant under translations $z \longrightarrow z - z_0$ we get that any shift $G_1(r, z-z_k)$ in the z-direction also satisfies equation (2.16). Due to the linearity of equation (2.16) we obtain a family of eigenfunctions

$$\hat{f}_N(r,z) = \sum_{k=1}^{N} a_k \frac{\sin\sqrt{r^2+(z-z_k)^2}}{\sqrt{r^2+(z-z_k)^2}}, \tag{2.18}$$

where a_k, z_k are arbitrary constants.

2.4. The linearity of the both equations (2.2) and (2.16) and their z-translational invariance yield that together with the exact solution $G_1(R)$ all its z-derivatives $\partial^n G_1(R)/\partial z^n$ of an arbitrary order n are solutions to the Helmholtz equation (2.16) and the corresponding vector fields $\mathbf{V}_{n+1}(\mathbf{x}) = \partial^n\mathbf{V}_1(\mathbf{x})/\partial z^n$ are solutions to the Beltrami equation (2.1). Using formulae (2.12) and (2.13) we get the following exact solutions to the Helmholtz and Beltrami equations:

$$f_2(\mathbf{x}) = \frac{\partial G_1}{\partial z} = zG_2, \quad \mathbf{V}_2(\mathbf{x}) = \frac{\partial \mathbf{V}_1(\mathbf{x})}{\partial z} = ((x+yz)G_3 + xz^2G_4)\hat{\mathbf{e}}_x + \tag{2.19}$$
$$((y-xz)G_3 + yz^2G_4)\hat{\mathbf{e}}_y + z(G_2 + 3G_3 + z^2G_4)\hat{\mathbf{e}}_z,$$

$$f_3(\mathbf{x}) = \frac{\partial^2 G_1}{\partial z^2} = G_2 + z^2 G_3, \quad \mathbf{V}_3(\mathbf{x}) = \frac{\partial^2 \mathbf{V}_1(\mathbf{x})}{\partial z^2} = (yG_3 + z(3x + yz)G_4 + xz^3 G_5)\hat{\mathbf{e}}_x + \quad (2.20)$$
$$(-xG_3 + z(3y - xz)G_4 + yz^3 G_5)\hat{\mathbf{e}}_y + (G_2 + (z+3)G_3 + z(z+5)G_4 + z^3 G_5)\hat{\mathbf{e}}_z.$$

The steady vector fields $\mathbf{V}_2(\mathbf{x})$ and $\mathbf{V}_3(\mathbf{x})$ together with the pressure $p_k(\mathbf{x}) = C - \rho|\mathbf{V}_k(\mathbf{x})|^2/2$ define exact solutions to the Euler equations (1.1). The same is true for their z-derivatives of an arbitrary order n.

Remark 5. The most detailed analysis of the force free magnetic fields $\mathbf{B}$ and Beltrami flows is presented in [8,19]. These works extensively use the Chandrasekhar, Kendall and Woltjer [9,10,37] general solution of the Beltrami equation in terms of the Bessel and Legendre functions. The exact solutions (2.19) and (2.20) are presented in terms of elementary functions and are absent in the works [8,19].

2.5. For any eigenfunction $f(r,z)$ satisfying equation (2.16) we consider the $\hat{\mathbf{e}}_z$-axisymmetric vector field $\mathbf{S}(\mathbf{x}) = f(r,z)\hat{\mathbf{e}}_z$ which satisfies equation $\Delta\mathbf{S}(\mathbf{x}) = -\mathbf{S}(\mathbf{x})$ in view of equation (2.16). Therefore we define vector field $\mathbf{V}_f(r,z)$ by formula analogous to (2.5). We have the identities $\operatorname{curl}(f(r,z)\hat{\mathbf{e}}_z) = -f_r\hat{\mathbf{e}}_\varphi$, $\operatorname{grad}\operatorname{div}(f(r,z)\hat{\mathbf{e}}_z) = f_{rz}\hat{\mathbf{e}}_r + f_{zz}\hat{\mathbf{e}}_z$. Substituting this into formula (2.5) we get $\mathbf{V}_f(r,z) = -f_r\hat{\mathbf{e}}_\varphi + f_{rz}\hat{\mathbf{e}}_r + (f_{zz} + f)\hat{\mathbf{e}}_z$. Transforming the last term here according to equation (2.16), we find

$$\mathbf{V}_f(r,z) = -f_r\hat{\mathbf{e}}_\varphi + \frac{1}{r}(rf_r)_z\hat{\mathbf{e}}_r - \frac{1}{r}(rf_r)_r\hat{\mathbf{e}}_z. \tag{2.21}$$

For the Cartesian coordinates x, y, z we have $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\cos\varphi - r\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sin\varphi$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\sin\varphi + r\frac{\mathrm{d}\varphi}{\mathrm{d}t}\cos\varphi$. Hence using formulae (2.15) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{e}}_r + r\frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{\mathbf{e}}_\varphi + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_z.$$

Therefore dynamical system (1.12), (2.21) takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial r}\right), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right), \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial f}{\partial r} \tag{2.22}$$

and describes dynamics along magnetic field lines.

It is evident that system (2.22) has first integral

$$H(r,z) = -r\frac{\partial f(r,z)}{\partial r}, \tag{2.23}$$

and therefore its dynamics occurs on invariant submanifolds which are defined by the equations $H(\mathbf{x}) = H = \text{const}$. The dynamical system (2.22) takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{r}\frac{\partial H}{\partial z}, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{r}\frac{\partial H}{\partial r}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r^2}H. \tag{2.24}$$

Remark 6. First integrals (2.23) corresponding to the exact solutions (2.19) and (2.20) are obtained from the first integral $H(r,z) = -r\partial G_1(R)/\partial z$ by the z-differentiation of the first and second order respectively. The same for the corresponding dynamical systems (2.24).

3. Invariant domains and equilibrium points

Let us consider the exact solution to the steady Euler equations (1.4) that is the vector field $\mathbf{V}_1(\mathbf{x})$ (2.14) with pressure $p(\mathbf{x}) = C - \rho|\mathbf{V}_1(\mathbf{x})|^2/2$. The axisymmetric form of this vector field is (2.21) with the eigenfunction $f = G_1(R) = (\sin R)/R$ (2.4). The function $H(r,z)$ (2.23) takes the form

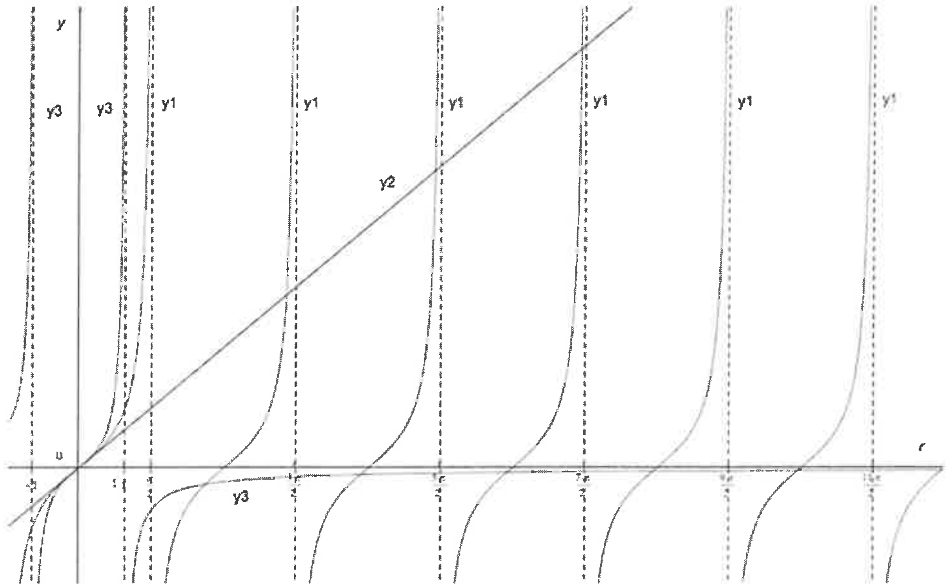

Fig. 2. Plots of functions $y_1 = \tan r$, $y_2 = r$ and $y_3 = r/(1 - r^2)$.

$$H(r, z) = -r^2 \frac{1}{R} \frac{\mathrm{d}G_1(R)}{\mathrm{d}R} = -r^2 G_2(R). \tag{3.1}$$

Using this formula and formulae (2.7), (2.11):

$$G_2(R) = \frac{1}{R} \frac{\mathrm{d}G_1(R)}{\mathrm{d}R} = \frac{1}{R^2} \left(\cos R - \frac{\sin R}{R} \right), \tag{3.2}$$
$$G_3(R) = \frac{1}{R} \frac{\mathrm{d}G_2(R)}{\mathrm{d}R} = \frac{1}{R^4} \left((3 - R^2) \frac{\sin R}{R} - 3 \cos R \right),$$

we derive

$$\frac{\partial H}{\partial z} = -r^2 \frac{\mathrm{d}G_2}{\mathrm{d}R} \frac{\partial R}{\partial z} = -r^2 z G_3, \tag{3.3}$$
$$\frac{\partial H}{\partial r} = -2rG_2 - r^2 \frac{\mathrm{d}G_2}{\mathrm{d}R} \frac{\partial R}{\partial r} = -2rG_2 - r^3 G_3.$$

Substituting formulae (3.3) into (2.24) we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{r} \frac{\partial H}{\partial z} = rzG_3(R), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{r} \frac{\partial H}{\partial r} = -2G_2(R) - r^2 G_3(R), \tag{3.4}$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r^2} H. \tag{3.5}$$

The function $H(r, z) = -r^2 G_2(R)$ is zero on the line $r = 0$ and on infinitely many semicircles $R = R_k$, $r \geq 0$ where $G_2(R_k) = 0$ that in view of (3.2) means

$$\tan R_k = R_k. \tag{3.6}$$

This equation has infinitely many solutions as follows from the plots of functions $y_1(r) = \tan r$ and $y_2(r) = r$, see Fig. 2. The equation $\tan R = R$ together with its smallest positive root "$kr_0 = 4.493$" first appeared in [28] where the term *"spheromak"* for a plasma equilibrium inside a sphere was first introduced.

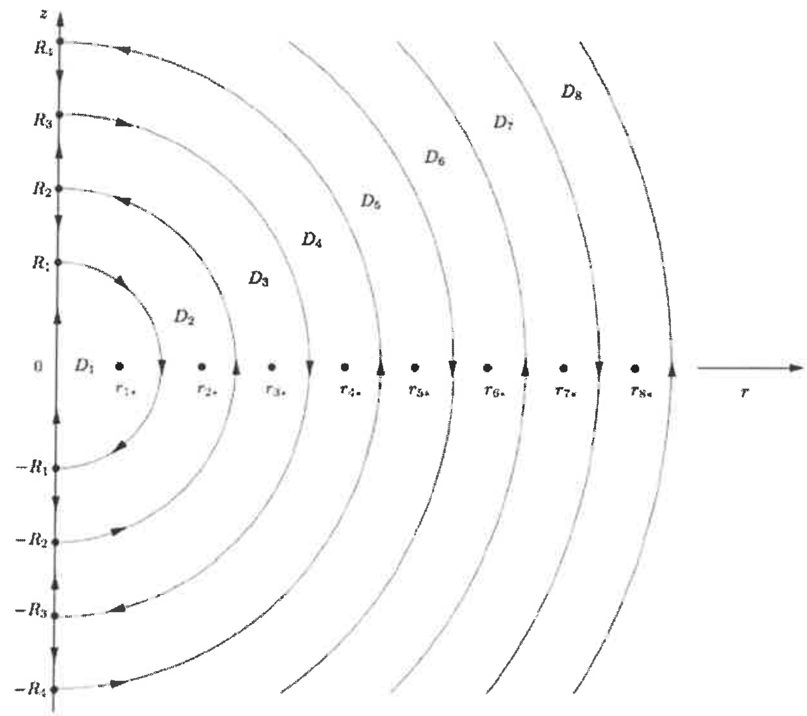

Fig. 3. Invariant domains $\mathcal{D}_n$ of dynamical system (3.4), its equilibrium points and separatrices.

Solutions R_k to equation (3.6) satisfy the relations

$$k\pi < R_k < (k + \frac{1}{2})\pi, \quad R_k \approx (k + \frac{1}{2})\pi, \quad k \longrightarrow \infty. \tag{3.7}$$

The first four numerical solutions are

$$R_1 \approx 4.4934, \quad R_2 \approx 7.7253, \quad R_3 \approx 10.9041, \quad R_4 \approx 14.0662. \tag{3.8}$$

These numbers coincide with the first four "values of $\lambda_k^{(1)}$ on the ball $B^3(1)$" presented in [8].

We consider the dynamical system (3.4) in the invariant domains $\mathcal{D}_1 : r^2 + z^2 < R_1^2$, $r > 0$ and $\mathcal{D}_n$: $R_{n-1}^2 < r^2 + z^2 < R_n^2$, $r > 0$, $n \geq 2$. Function $H(r, z) = -r^2 G_2(R)$ is zero on the boundary of each domain $\mathcal{D}_n$, see Fig. 3.

Lemma 2. *Dynamical system (3.4) in each invariant domain $\mathcal{D}_n$ has only one equilibrium point $(r_{n*}, z = 0)$ that is the point of a non-degenerate maximum of function $H(r, z)$ in each domain $\mathcal{D}_{2k+1}$ and the point of its non-degenerate minimum in each domain $\mathcal{D}_{2k}$. Function $H(r, z) > 0$ in domains $\mathcal{D}_{2k+1}$ and $H(r, z) < 0$ in domains $\mathcal{D}_{2k}$. All equilibrium points $(r_{n*}, z = 0)$ are centers and all trajectories of system (3.4) in each invariant domain $\mathcal{D}_n$ are closed curves.*

Proof. Let us find values of function $G_3(R)$ at the points R_n. Substituting $G_2(R_n) = 0$ into formula (2.11) we get

$$G_3(R_n) = -\frac{\sin R_n}{R_n^3} = -\operatorname{sign}(\sin R_n)\frac{1}{R_n^2\sqrt{R_n^2 + 1}} \neq 0. \tag{3.9}$$

Here we used identity $\sin^2 x = \tan^2 x(\tan^2 x + 1)^{-1}$ and equation (3.6). Hence functions $G_2(R)$ and $G_3(R)$ cannot be zero simultaneously. Therefore equations (3.4) yield for their equilibrium points: $z = 0$ and

$2G_2(r) + r^2G_3(r) = 0$. The latter equation by virtue of formulae (3.2) is $\sin r(1-r^2)/r - \cos r = 0$ and has the equivalent form

$$\tan r = \frac{r}{1-r^2}. \tag{3.10}$$

The plots of functions $y_1(r) = \tan r$ and $y_3(r) = r/(1-r^2)$ in Fig. 2 clearly show that equation (3.10) has infinitely many solutions

$$(k - \frac{1}{2})\pi < r_{k*} < k\pi, \quad r_{k*} \approx k\pi, \quad k \longrightarrow \infty. \tag{3.11}$$

Comparing formulae (3.7) and (3.11) we see that in each invariant domain $\mathcal{D}_k$ there is only one equilibrium point $(r_{k*},\ z_{k*} = 0)$. The invariant domains $\mathcal{D}_n$ and equilibrium points $(r_{n*}, z = 0)$ are shown in Fig. 3.

At the equilibrium points $r_* = R_*,\ z_* = 0$, we find from equations (3.1), (3.2)

$$H(r_*, z_*) = \frac{\sin r_*}{r_*} - \cos r_* = \frac{\sin r_*}{r_*}\left(1 - \frac{r_*}{\tan r_*}\right).$$

Substituting here equation (3.10) we get $H(r_*, z_*) = r_* \sin r_*$. Using identity $\sin r = \mathrm{sign}(\sin r)|\tan r|(1 + \tan^2 r)^{-1/2}$ and substituting (3.10) we get

$$H(r_*, z_*) = H_* = r_* \sin r_* = \mathrm{sign}(\sin r_*)\frac{r_*^2}{\sqrt{r_*^4 - r_*^2 + 1}}. \tag{3.12}$$

Using formulae (3.1) and (3.3) we derive

$$\frac{\partial H}{\partial r} = \frac{2}{r}H - r^3G_3 = \frac{2}{r}H - \frac{r^3}{R}\frac{\mathrm{d}G_2}{\mathrm{d}R}, \tag{3.13}$$

$$\frac{\partial^2 H}{\partial r^2} = -\frac{2}{r^2}H + \frac{2}{r}\frac{\partial H}{\partial r} - \frac{3r^2}{R}\frac{\mathrm{d}G_2}{\mathrm{d}R} + \frac{r^4}{R^3}\frac{\mathrm{d}G_2}{\mathrm{d}R} - \frac{r^4}{R^2}\frac{\mathrm{d}^2G_2}{\mathrm{d}R^2}, \tag{3.14}$$

$$\frac{\partial^2 H}{\partial z\partial r} = -z\frac{\partial(r^2G_3)}{\partial r}, \quad \frac{\partial^2 H}{\partial z^2} = -r^2G_3 - z\frac{\partial(r^2G_3)}{\partial z}. \tag{3.15}$$

At the equilibrium points $(r_*, z_* = 0)$ we have $\partial H/\partial r = 0$. Hence from (3.13) we find

$$r_*^2G_3 = r_*\frac{\mathrm{d}G_2}{\mathrm{d}R} = \frac{2}{r_*^2}H_*. \tag{3.16}$$

Substituting formula (3.16) into equation (3.14) and using $r_* = R_*$ we get

$$\frac{\partial^2 H}{\partial r^2} = -\frac{6}{r^2}H_* - r_*^2\frac{\mathrm{d}^2G_2}{\mathrm{d}R^2}. \tag{3.17}$$

Substituting into (3.17) $\mathrm{d}^2G_2/\mathrm{d}R^2$ from equation (2.9) and using equations (3.1) and (3.16) we find

$$\frac{\partial^2 H}{\partial r^2} = -\frac{6}{r_*^2}H_* + r_*^2\left(\frac{4}{R_*}\frac{\mathrm{d}G_2}{\mathrm{d}R} + G_2\right) = \frac{2}{r_*^2}H_* - H_* = \frac{(2-r_*^2)}{r_*^2}H_*. \tag{3.18}$$

From equations (3.15) and (3.16) we find at $z_* = 0$:

$$\frac{\partial^2 H}{\partial z\partial r} = 0, \quad \frac{\partial^2 H}{\partial z^2} = -r^2G_3 = -\frac{2}{r_*^2}H_*. \tag{3.19}$$

For all solutions r_* to the equation (3.10) we have $r_*^2 > 2$. Hence formulae (3.18), (3.19) and (3.12) prove that the equilibrium points $(r_*, z_* = 0)$ of system (6.3) are non-degenerate local minima of function $H(r,z)$ if $H(r_*, 0) = H_* = r_* \sin r_* < 0$ and local maxima if $H_* = r_* \sin r_* > 0$.

Formulae (3.1) and (2.8), $G_2(0) = -1/3$, show that $H(r,z) > 0$ in the domain $\mathcal{D}_1$. Since $R_1, R_2, R_3, \cdots$ are subsequent zeroes of function $G_2(R)$ and $\mathrm{d}G_2/\mathrm{d}R(R_n) = R_n G_3(R_n) \neq 0$ (3.9) we get that function $G_2(R)$ changes its sign near each zero $R = R_n$. The same is evidently true for function $H(r,z) = -r^2 G_2(R)$ (3.1). Since boundary of each domain $\mathcal{D}_n$ contains two semicircles $R = R_n$ and $R = R_{n-1}$, we find by induction that $H(r,z) > 0$ in domains $\mathcal{D}_{2k+1}$ and $H(r,z) < 0$ in domains $\mathcal{D}_{2k}$.

Since function $H(r,z)$ does not have other critical points inside $\mathcal{D}_n$ we find that all curves of constant level of $H(r,z)$ are closed curves C_H, $H = \text{const}$. Since $H(r,z)$ is a first integral of system (3.4) we get that all trajectories of the system in invariant domain $\mathcal{D}_n$ are closed curves C_H that go around the equilibrium point $(r_{n*}, z = 0)$. Therefore all equilibrium points $(r_{n*}, z = 0)$ are centers. □

Lemma 3. *All equilibrium points $(r = 0, z = \pm R_n)$ are saddles. The semicircles $R = R_n$ are separatrices of dynamical system (3.4) which go down from the equilibrium points $(r = 0, z = R_n)$ to the equilibrium points $(r = 0, z = -R_n)$ if $n = 2k+1$ and in the opposite direction if $n = 2k$.*

Proof. At the equilibrium points $(r = 0, z = \sigma R_n)$, $\sigma = \pm 1$, dynamical system (3.4) has the eigenvalues

$$\lambda_r = \sigma R_n G_3(R_n), \quad \lambda_z = -2\sigma R_n G_3(R_n). \tag{3.20}$$

Therefore (since $G_3(R_n) \neq 0$, (3.9)), all equilibrium points $(r = 0, z = \sigma R_n)$ are saddles. Their separatrices are the semicircles $R = R_n$ and open intervals on the axis z, $r = 0$ between these equilibrium points, see Fig. 3.

As above, since $R_1, R_2, R_3, \cdots$ are subsequent zeroes of function $G_2(R)$ and $G_2(0) = -1/3$, we find from formulae $G_3(R) = R^{-1}\mathrm{d}G_2(R)/\mathrm{d}R$ (3.2) and $G_3(R_n) \neq 0$ (3.9) that $G_3(R_1) > 0$, $G_3(R_{2k}) < 0$, $G_3(R_{2k+1}) > 0$. Hence formulae (3.20) yield $\lambda_r(r = 0, z = -R_{2k+1}) < 0$, $\lambda_r(r = 0, z = R_{2k+1}) > 0$, $\lambda_r(r = 0, z = -R_{2k}) > 0$, $\lambda_r(r = 0, z = R_{2k}) < 0$. Hence dynamics on the semicircles $R = R_{2k+1}$ is down and dynamics on the semicircles $R = R_{2k}$ is up, see Fig. 3. □

Corollary 1. *Rotation along the closed trajectories in the invariant domains $\mathcal{D}_{2k+1}$ is clockwise and is counter-clockwise in the domains $\mathcal{D}_{2k}$.*

Proof. Boundary of each domain $\mathcal{D}_n$ contains two semicircles $R = R_n$ and $R = R_{n-1}$. Dynamics on these separatrices is shown in Fig. 3. Hence by the continuity we get the clockwise rotation along closed trajectories in domains $\mathcal{D}_{2k+1}$ and the counter-clockwise rotation along trajectories in domains $\mathcal{D}_{2k}$. □

4. A method for finding the moduli spaces of vortex knots

4.1. We call a dynamical system in an invariant domain $\mathcal{D} \subset \mathbb{R}^3$ *non-degenerately integrable* if its trajectories are either quasi-periodic or periodic curves on the invariant tori $\mathbb{T}^2 \subset \mathbb{R}^3$ and topology of trajectories is not the same for all tori that means the topology is changing from one torus to another. Lemma 2 yields that all invariant submanifolds $H(r,z) = \text{const.} \neq 0$ of system (3.4)–(3.5) are tori $\mathbb{T}_H^2 = C_H \times S^1 \subset \mathbb{R}^3$, where circle S^1 corresponds to the angular variable φ.

To prove that topology of trajectories on the tori $\mathbb{T}_H^2$ is changing from one torus to another we choose another time variable τ that simplifies the analysis:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{H}{2\pi r^2}. \tag{4.1}$$

It is evident that topology of trajectories does not depend on their parametrization and is the same in time t and time τ. In the new time τ, the dynamical system (2.24) takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\frac{2\pi r}{H}\frac{\partial H}{\partial z}, \qquad \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{2\pi r}{H}\frac{\partial H}{\partial r}, \tag{4.2}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = 2\pi. \tag{4.3}$$

The main advantage of the time change (4.1) is that the τ-derivative (4.3) of the angular variable φ is a constant equal to 2π.

System (4.2) evidently is a reparametrized Hamiltonian system with Hamiltonian function $H(r, z) = -rf_r(r, z)$. The invariant submanifolds of system (2.24) defined by equation $H(r, z) = 0$, namely all semi-circles $R = R_n$ and the axis z, $r = 0$, become the singular subsets of system (4.2).

Corollary 2. *All trajectories of the dynamical system (4.2) are closed curves and their rotation around the equilibrium points $(r_{n*}, 0)$ is clockwise.*

Proof. Applying Lemma 2 we get that all trajectories of system (4.2) in any invariant domain $\mathcal{D}_n$ are closed curves that go around the equilibrium point $(r_{n*}, 0)$. The time change (4.1) preserves direction of time in invariant domains $\mathcal{D}_{2k+1}$ where $H(r, z) > 0$ and reverses it in the domains $\mathcal{D}_{2k}$ where $H(r, z) < 0$. Therefore using Corollary 1 we get that the rotation along all closed trajectories of system (4.2) is clockwise. □

4.2. Lemma 2 yields that in each invariant domain $\mathcal{D}_k$ all trajectories of system (4.2) are closed curves C_H: $H(r, z) = H$. For each invariant domain $\mathcal{D}_k$ we define a function $\tau_k(H)$ equal to the minimal period of trajectory $C_H \subset \mathcal{D}_k$ of system (4.2). Functions $\tau_k(H)$ are defined in different domains $0 \leq |H| \leq |H|_{k*}$ and have different ranges. Since trajectories C_H continuously depend on parameter H and system (4.2) is smooth in the interior of each domain $\mathcal{D}_k$ we get from the general theory of dynamical systems [15] that functions $\tau_k(H)$ are continuous.

Trajectories of system (4.2)–(4.3) move on invariant tori $\mathbb{T}^2_H = C_H \times S^1$ in the 3-dimensional space r, z, φ where the circle S^1 corresponds to the angular variable $\varphi \mathrm{mod}(2\pi)$.

Proposition 1. *Topology of trajectories is changing from one torus $\mathbb{T}^2_{H_1}$ to another $\mathbb{T}^2_{H_2}$ if and only if the function of periods $\tau(H)$ is not constant.*

Proof. If the continuous function $\tau(H) \neq$ const. then it takes all rational and all irrational values in some interval (a, b).

Let a closed trajectory C_{H_1} have a rational period $\tau(H_1) = p/q$. During the time $\tau(H_1)$ the angular variable φ is changed for $2\pi\tau(H_1)$ because $\mathrm{d}\varphi/\mathrm{d}\tau = 2\pi$. After q complete turns of trajectory around the closed curve C_{H_1} the angular variable φ is changed for $q(2\pi\tau(H_1)) = 2\pi p$, because $\tau(H_1) = p/q$. Hence all trajectories on the torus $\mathbb{T}^2_{H_1}$ are closed curves.

Now let a closed trajectory C_{H_2} have an irrational period $\tau(H_2)$. Then after any N complete turns of trajectory around the closed curve C_{H_2} the angular variable φ is changed for $2\pi N\tau(H_2)$. For any integers N and M we have $2\pi N\tau(H_2) \neq 2\pi M$ because $\tau(H_2)$ is irrational, $\tau(H_2) \neq M/N$. Hence all trajectories on the torus $\mathbb{T}^2_{H_2}$ are non-closed infinite curves which are dense on $\mathbb{T}^2_{H_2}$.

Hence topology of trajectories is changing from one torus to another if function of periods $\tau(H)$ is not constant.

If $\tau(H) = \tau_1 =$ const. then trajectories on all tori either are all closed (if τ_1 is rational) or are all dense (if τ_1 is irrational). This means that if function $\tau(H)$ is constant then all trajectories have the same topology. □

Structure of knots:

All trajectories on the torus $\mathbb{T}^2_H$ with the period $\tau(H) = p/q$ (p and q are coprime) are closed curves which make q complete turns around the meridians and p complete turns around the longitudes of the torus $\mathbb{T}^2_H$. Hence they form a torus knot $K_{p,q}$.

For example, all trajectories on $\mathbb{T}^2_H$ with the period $\tau(H) = 3/4$ make 4 turns around the meridians and 3 turns around the longitudes. They form a non-trivial torus knot $K_{3,4}$ that is shown in Fig. 8 below.

A torus knot $K_{p,q}$ and its mirror image $\tilde{K}_{p,q}$ have opposite directions of rotation around the meridians.

Corollary 3. *If for some integers p and q a torus knot $K_{p,q}$ is realized by vortex lines for the spheromak fluid flow then its mirror image $\tilde{K}_{p,q}$ is not realized.*

Proof. Indeed, by Corollary 2 all closed trajectories of dynamical system (4.2) rotate in the clockwise direction, so the opposite (counter-clockwise) rotation is not realized. □

4.3. The main method

To construct the moduli space $\mathcal{S}(\mathcal{D})$ of vortex knots it is necessary to find the ranges of all functions of periods $\tau_k(H)$ for the invariant domains $\mathcal{D}_k$ for $k = 1, 2, 3, \cdots$. Indeed, using the proof of Proposition 1 and Corollary 3 we get that all rational numbers p/q from those ranges define all torus knots $K_{p,q}$ realized by vortex lines for the spheromak flow. To find the ranges of the continuous functions $\tau_k(H)$ we calculate their limits

$$\lim_{H \longrightarrow 0} \tau_k(H), \qquad \lim_{H \longrightarrow H_{k*}} \tau_k(H), \tag{4.4}$$

where H_{k*} are the values of function $H(r, z)$ at the equilibrium points (r_{k*}, z_{k*}). Then since the limits occur to be different and the functions $\tau_k(H)$ occur to be monotonous in their domains we get their ranges between the above limits. This leads to the construction of the moduli spaces $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}_m(\mathbb{B}^3_a)$ in Sections 8 and 9.

5. Limits of functions $\tau_k(H)$ at $H \longrightarrow H_{k*}$

Lemma 4. *Functions of periods $\tau_k(H)$ have the following limits at the equilibrium points $(r_{k*}, z_{k*} = 0)$ in the invariant domains $\mathcal{D}_k$:*

$$\lim_{H \longrightarrow H_{k*}} \tau(H) = \tau_k(H_{k*}) = \frac{r_{k*}}{\sqrt{2(r^2_{k*} - 2)}}. \tag{5.1}$$

Proof. System (4.2) implies that all equilibrium points with $r \neq 0$ and $H \neq 0$ satisfy equations $\partial H/\partial z = 0$, $\partial H/\partial r = 0$. Hence all equilibrium points with $r \neq 0$ are extrema of the function $H(r, z)$. We have shown in Lemma 2 of Section 3 that these extrema necessarily are either non-degenerate local minima or non-degenerate local maxima of function $H(r, z)$ and that in each domain $\mathcal{D}_k$ there is only one local minimum or local maximum.

Dynamics of any nonlinear dynamical system $\dot{r} = f_1(r, z)$, $\dot{z} = f_2(r, z)$ near its non-degenerate equilibrium point $a(r_0, z_0)$ is approximated by the linear system in variations [15]:

$$\frac{\mathrm{d}\delta r}{\mathrm{d}\tau} = \frac{\partial f_1}{\partial r}(a)\delta r + \frac{\partial f_1}{\partial z}(a)\delta z, \quad \frac{\mathrm{d}\delta z}{\mathrm{d}\tau} = \frac{\partial f_2}{\partial r}(a)\delta r + \frac{\partial f_2}{\partial z}(a)\delta z, \tag{5.2}$$

where $\delta r(t) = r(t) - r_0$, $\delta z(t) = z(t) - z_0$. For system (4.2) we have at the equilibrium points $a_k(r = r_{k*}$, $z = 0)$:

$$\frac{\partial f_1}{\partial r} = 0, \quad \frac{\partial f_1}{\partial z} = -\frac{2\pi r_{k*}}{H_{k*}}\frac{\partial^2 H}{\partial z^2}, \quad \frac{\partial f_2}{\partial r} = \frac{2\pi r_{k*}}{H_{k*}}\frac{\partial^2 H}{\partial r^2}, \quad \frac{\partial f_2}{\partial z} = 0.$$

Hence the corresponding system in variations (5.2) yields

$$\frac{\mathrm{d}^2\delta r}{\mathrm{d}\tau^2} = -\left(\frac{2\pi r_{k*}}{H_{k*}}\right)^2 \frac{\partial^2 H}{\partial r^2}(a_k)\frac{\partial^2 H}{\partial z^2}(a_k)\delta r.$$

Substituting here the values of partial derivatives (3.18) and (3.19) we get

$$\frac{\mathrm{d}^2\delta r}{\mathrm{d}\tau^2} = -8\pi^2\frac{r_{k*}^2 - 2}{r_{k*}^2}\delta r. \tag{5.3}$$

All solutions to equation (5.3) have the form $\delta r(\tau) = A\sin(2\pi\omega_k\tau + c)$ where $\omega_k = \sqrt{2(r_{k*}^2 - 2)}/r_{k*}$. The solutions evidently are periodic with period $T_k = 1/\omega_k$. Since the limit of the periods of oscillations of a non-linear system near its non-degenerate center is equal to the period of oscillations of the corresponding system in variations [15] we get the equation (5.1). □

Remark 7. Since in each invariant domain $\mathcal{D}_k$ there is only one point of extrema $(r_{k*}, z_{k*} = 0)$ we obtain that this point is a global minimum or maximum of function $H(r,z)$ for the whole domain $\mathcal{D}_k$. Hence we get that all curves C_H of constant levels $H(r,z) = \text{const.}$ of function H consist of one component. There is a one to one correspondence between the curves $C_H \subset \mathcal{D}_k$ and values of function $H(r,z)$ in domain $\mathcal{D}_k$: different curves correspond to different values of function H. For the complete system (4.2)–(4.3) we conclude that: *(a)* constant levels of first integral $H(r,z) = \text{const.} \neq 0$ define invariant tori $\mathbb{T}_H^2 = C_H \times S^1 \subset \mathcal{D}_k \times S^1 \subset \mathbb{R}^3$ and *(b)* different tori in $\mathcal{D}_k \times S^1$ correspond to different values of $H(r,z)$.

Remark 8. The first four positive numeric solutions to equation (3.10) are

$$r_{1*} \approx 2.7437, \quad r_{2*} \approx 6.1168, \quad r_{3*} \approx 9.3166, \quad r_{4*} \approx 12.4859. \tag{5.4}$$

Using numerical values (5.4) we find values of function $H(r_{k*}, z_{k*})$ (3.12) at the first four equilibrium points $(r_{k*}, z_{k*} = 0)$:

$$H_{1*} \approx 1.0631, \quad H_{2*} \approx -1.0131, \quad H_{3*} \approx 1.0059, \quad H_{4*} \approx -1.0037.$$

Formulae (3.11), (3.12) yield that function $|H(r_{k*}, 0)|$ is monotonously decreasing to its limit 1 when $k \longrightarrow \infty$.

Remark 9. Using numerical values of r_{k*} (5.4) we find from (5.1) the first four values of the limit periods $\tau_k(H_{k*})$

$$\tau_1(H_{1*}) \approx 0.8252, \ \tau_2(H_{2*}) \approx 0.7268, \ \tau_3(H_{3*}) \approx 0.7154, \ \tau_4(H_{4*}) \approx 0.7117. \tag{5.5}$$

Function $r/\sqrt{2(r^2-2)}$ is monotonously decreasing from ∞ at $r = \sqrt{2}$ to $1/\sqrt{2} \approx 0.7071$ when $r \longrightarrow \infty$. Therefore the limit periods $\tau_k(H_{k*}) = r_{k*}/\sqrt{2(r_{k*}^2 - 2)}$ (5.1) have different values at the different critical points $(r_{k*}, z_{k*} = 0)$. Evidently formula (5.1) yields

$$\lim_{k \longrightarrow \infty} \tau_k(H_{k*}) = \frac{1}{\sqrt{2}} \tag{5.6}$$

because $r_{k*} \longrightarrow \infty$ at $k \longrightarrow \infty$, see (3.11) and Fig. 3.

6. Limits of functions $\tau_k(H)$ at $H \longrightarrow 0$

Lemma 5. *Functions of periods have the following limits in domains $\mathcal{D}_k$:*

$$\mathcal{D}_1 : \quad \lim_{H \longrightarrow 0} \tau_1(H) = p_1 = \frac{1}{2\pi} R_1. \tag{6.1}$$

$$\mathcal{D}_k,\ k \geq 2 : \quad \lim_{H \longrightarrow 0} \tau_k(H) = p_k = \frac{1}{2\pi}(R_k - R_{k-1}). \tag{6.2}$$

Proof. Dynamical system (3.4)–(3.5) after the time change (4.1) takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -2\pi r z \frac{G_3(R)}{G_2(R)}, \qquad \frac{\mathrm{d}z}{\mathrm{d}\tau} = 4\pi + 2\pi r^2 \frac{G_3(R)}{G_2(R)}, \tag{6.3}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = 2\pi. \tag{6.4}$$

Formula (3.9) yields $G_3(R_k) \neq 0$. Hence system (6.3) is singular at the semi-circles $r^2 + z^2 = R_k^2$, $r \geq 0$ on which $G_2(R_k) = 0$.

System (6.3) evidently has invariant line $L : r = 0$ on which the system takes the form

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}z}{\mathrm{d}\tau} = 4\pi. \tag{6.5}$$

For the absolute value $V(r, z)$ of the vectors (6.3) we find

$$\frac{1}{4\pi^2} V^2(r, z) = \frac{1}{4\pi^2}\left(\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\right) = r^2 z^2 \frac{G_3^2}{G_2^2} + 4 + 4r^2 \frac{G_3}{G_2} + r^4 \frac{G_3^2}{G_2^2} = \tag{6.6}$$
$$\frac{r^2}{G_2^2}(z^2 G_3^2 + 4G_2 G_3) + 4 + r^4 \frac{G_3^2}{G_2^2}.$$

At the points $P_{k\pm} : r = 0, z = \pm R_k$, $R = \sqrt{r^2 + z^2} = R_k$ we get from (3.9)

$$z^2 G_3^2 + 4G_2 G_3 = \frac{1}{R_k^2(R_k^2 + 1)} > 0. \tag{6.7}$$

Hence in an ε-neighborhood of each point $P_{k\pm}$ we get from (6.6), (6.7): $V \geq 4\pi$.

Function $H(r, z) = -r^2 G_2(R)$ is zero on the boundary of each domain $\mathcal{D}_k$. When first integral $H(r, z)$ tends to zero, the trajectory $C_H : |H(r, z)| = c_0 < \varepsilon^2$ in an invariant domain $\mathcal{D}_k$ is in $C\varepsilon$-neighborhood of the boundary of this domain. Boundary of domain $\mathcal{D}_1$ consists of the segment $I_1 : r = 0, -R_1 \leq z \leq R_1$ and the semi-circle $S_1 : r^2 + z^2 = R_1^2$, $r \geq 0$. Trajectory C_H at $H \longrightarrow 0$ moves near the boundary $\partial\mathcal{D}_1 = I_1 \bigcup S_1$ and consists of an arc A_1 of length $\approx 2R_1$ near the segment I_1, two small arcs $B_{1\pm}$ of length ε near points $P_{1\pm}$ and an arc C_1 of length $\approx \pi R_1$ near the semicircle S_1. For the speed V (6.6) of dynamics along these arcs we have from (6.5) and (6.6)–(6.7) at $H \longrightarrow 0$:

$$V_{A_1} \longrightarrow 4\pi, \quad V_{B_{1\pm}} \geq 4\pi, \quad \varepsilon \longrightarrow 0, \quad V_{C_1} \longrightarrow \infty. \tag{6.8}$$

The period $\tau_1(H)$ of trajectory C_H is equal to the total time of dynamics around the closed curve C_H that is

$$\tau_1(H) = \frac{2R_1}{V_{A_1}} + \frac{\varepsilon}{V_{B_{1+}}} + \frac{\varepsilon}{V_{B_{1-}}} + \frac{\pi R_1}{V_{C_1}}. \tag{6.9}$$

Using formulae (6.8) we find that function $\tau_1(H)$ (6.9) has the limit value (6.1) at $H \longrightarrow 0$.

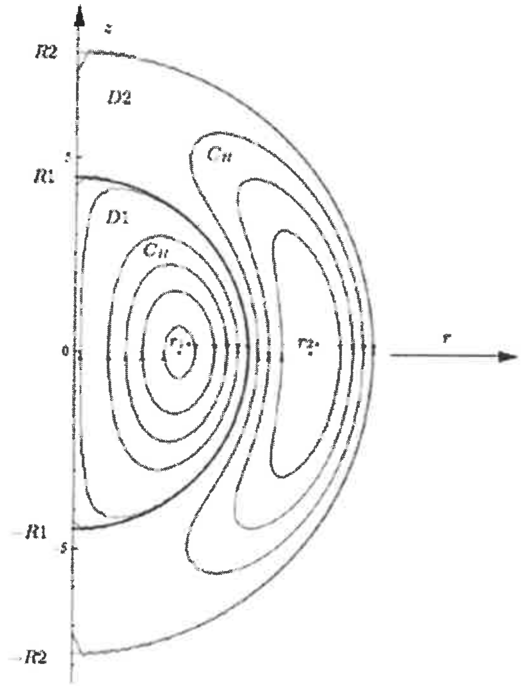

Fig. 4. Phase portrait of dynamical system (4.2) in the invariant domains $\mathcal{D}_1$, $\mathcal{D}_2$. All rotations are clockwise.

Boundary of domain $\mathcal{D}_k$ consists of two segments $I_{k.+} : r = 0, R_{k-1} \leq z \leq R_k$ and $I_{k.-} : r = 0, -R_k \leq z \leq -R_{k-1}$, and two semicircles $S_k : r^2 + z^2 = R_k^2, r \geq 0$ and $S_{k-1} : r^2 + z^2 = R_{k-1}^2, r \geq 0$. Trajectory C_H of system (6.3) in the domain $\mathcal{D}_k$ at $H \longrightarrow 0$ moves near its boundary $\partial\mathcal{D}_k = I_{k+} \bigcup I_{k-} \bigcup S_k \bigcup S_{k-1}$ and consists of two arcs $A_{k.\pm}$ of length $\approx (R_k - R_{k-1})$ near the segments $I_{k.\pm}$, four small arcs $B_{k.\pm}$ and $B_{k-1.\pm}$ of length ε near four points $P_{k.\pm}$ and $P_{k-1.\pm}$ and arcs C_k of length $\approx \pi R_k$ and C_{k-1} of length $\approx \pi R_{k-1}$ near the semicircles S_k and S_{k-1}, see Figs. 3 and 4. For the speed of trajectory along these arcs we find from (6.6), (6.7) at $H \longrightarrow 0$:

$$V_{A_{k.\pm}} \longrightarrow 4\pi, \quad V_{B_{k.\pm}}, V_{B_{k-1.\pm}} \geq 4\pi, \quad \varepsilon \longrightarrow 0, \quad V_{C_k}, V_{C_{k-1}} \longrightarrow \infty. \tag{6.10}$$

The period $\tau_k(H)$ of trajectory C_H is equal to the total time of dynamics around the closed curve C_H that is

$$\tau_k(H) = \frac{R_k - R_{k-1}}{V_{A_{k.+}}} + \frac{R_k - R_{k-1}}{V_{A_{k.-}}} + \frac{2\varepsilon}{V_{B_{k.\pm}}} + \frac{2\varepsilon}{V_{B_{k-1.\pm}}} + \frac{\pi R_k}{V_{C_k}} + \frac{\pi R_{k-1}}{V_{C_{k-1}}}. \tag{6.11}$$

Using formulae (6.10) we find that function $\tau_k(H)$ (6.11) has the limit value (6.2) at $H \longrightarrow 0$. □

Remark 10. Using numerical values (3.8) we find the first four values of p_k:

$$p_1 \approx 0.7151, \quad p_2 \approx 0.5144, \quad p_3 \approx 0.5062, \quad p_4 \approx 0.5033. \tag{6.12}$$

From equation (3.6) and Fig. 2 we see that $p_k > 1/2$ and p_k (6.2) is a monotonously decreasing function of k. In view of (3.7) we get

$$\lim_{k \longrightarrow \infty} p_k = \lim_{k \longrightarrow \infty} \frac{1}{2\pi}(R_k - R_{k-1}) = \frac{1}{2}. \tag{6.13}$$

Numerical values (6.12) show that the convergence to the limit 1/2 (6.13) is rather fast.

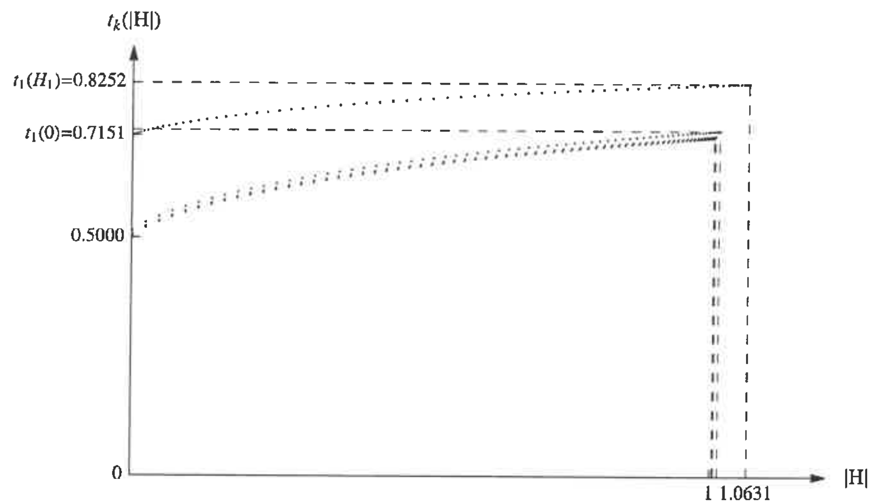

Fig. 5. Numerical calculations of functions of periods $\tau_1(H)$, $\tau_2(|H|)$, $\tau_3(H)$ and $\tau_4(|H|)$.

7. Dynamics of vortex lines

Proposition 2. *For the spheromak solution to the steady Euler equations*

$$\mathbf{V}_1(x,y,z) = (yG_2 + xzG_3)\hat{\mathbf{e}}_x + (-xG_2 + yzG_3)\hat{\mathbf{e}}_y + (G_1 + G_2 + z^2G_3)\hat{\mathbf{e}}_z, \quad (7.1)$$
$$p = C - \rho_c|\mathbf{V}_1(x,y,z)|^2/2,$$

the dynamics of its vortex lines is non-degenerately integrable in each invariant domain $\mathcal{D}_k \times S^1$, $k = 1, 2, \cdots$. *Here* $R = \sqrt{x^2+y^2+z^2}$ *and analytic functions* $G_1(R)$, $G_2(R)$, $G_3(R)$ *have form (2.4) and (3.2).*

Proof. The vector field $\mathbf{V}_1(x,y,z)$ (7.1) coincides with (2.14) and has the z-axisymmetric form (2.21) with $f(r,z) = G_1(r,z) = \sin(R)/R$. Hence dynamics along the vortex lines $\mathrm{d}\mathbf{x}/\mathrm{d}t = \operatorname{curl}\mathbf{V}_1(\mathbf{x}) = \mathbf{V}_1(\mathbf{x})$ occurs on the $\hat{\mathbf{e}}_z$-axisymmetric tori $\mathbb{T}^2_H$ defined by equations $H(\mathbf{x}) = -(x^2+y^2)G_2(R) = H = \text{const}$.

The results obtained in Sections 5 and 6 prove that in each invariant domain $\mathcal{D}_k$ the functions of periods $\tau_k(H)$ are continuously changing between the limits $p_k = (R_k - R_{k-1})/2\pi$ at $H \longrightarrow 0$ and the limits $\tau_k(H_{k*}) = r_{k*}/\sqrt{2(r^2_{k*}-2)}$ at $H \longrightarrow H_{k*}$. To establish the behavior of the functions $\tau_k(H)$ between these two limits we have calculated the functions numerically for $k = 1,2,3,4$. The results of the numerical calculations are shown in Fig. 5 which evidently demonstrates that the functions $\tau_k(|H|)$ are monotonous. Hence when $0 < |H| < |H_{k*}|$, the functions are changing monotonously in the following ranges

$$p_k = \frac{1}{2\pi}(R_k - R_{k-1}) < \tau_k(H) < \frac{r_{k*}}{\sqrt{2(r^2_{k*}-2)}} = \tau_k(H_{k*}), \quad (7.2)$$

where $p_1 = R_1/(2\pi)$. Since the highest lower bound p_k (6.2) is monotonously decreasing to the limit 1/2 (6.13) when $k \longrightarrow \infty$ and function $\tau_k(H_{k*})$ is monotonously decreasing to its limit $1/\sqrt{2} \approx 0.7071$ (5.6) and $1/2 < 1/\sqrt{2}$, we obtain that for each k the limits p_k and $\tau_k(H_{k*})$ in (7.2) are different for all invariant domains $\mathcal{D}_k$. Hence for each domain $\mathcal{D}_k$ the continuous function $\tau_k(H)$ is not constant.

Applying Proposition 1 we conclude that dynamical system (6.3)–(6.4) is non-degenerately integrable in each invariant domain $\mathcal{D}_k \times S^1$. □

Remark 11. The plots of continuous functions of periods $\tau_k(H)$ for the invariant domains $\mathcal{D}_k$ are shown in Fig. 6 for $k = 1$ and in Fig. 7 for $k \geq 2$. For the first four invariant domains $\mathcal{D}_k$ we find from the formulae (6.12) and (5.5):

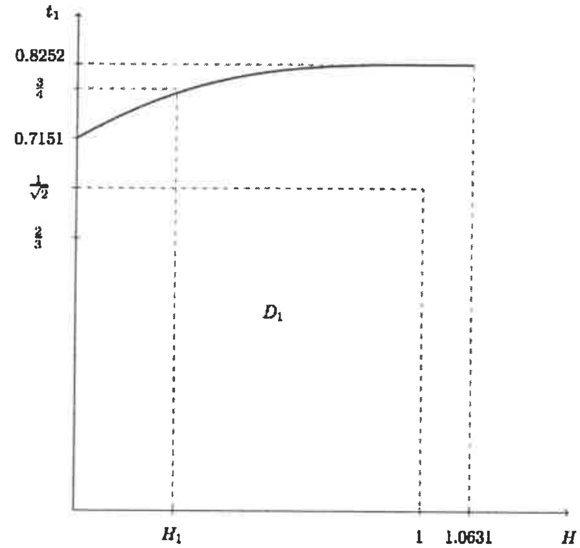

Fig. 6. Function of periods $\tau_1(H)$ for invariant domain $\mathcal{D}_1$.

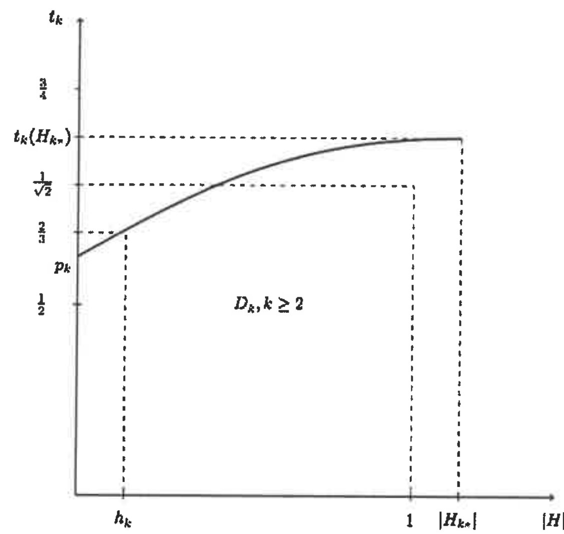

Fig. 7. Function of periods $\tau_k(H)$ for invariant domain $\mathcal{D}_k$, $k \geq 2$.

$$\begin{aligned} &\mathcal{D}_1 : \ 0.7152 < \tau_1(H) < 0.8252, \quad \mathcal{D}_2 : \ 0.5144 < \tau_2(H) < 0.7268, \\ &\mathcal{D}_3 : \ 0.5062 < \tau_3(H) < 0.7154, \quad \mathcal{D}_4 : \ 0.5033 < \tau_4(H) < 0.7117. \end{aligned} \tag{7.3}$$

The formulae (7.3) for domain $\mathcal{D}_1$ yield that in the first invariant ball $\mathbb{B}_{a_1}$ the safety factor $\mathbf{q}(H) = \tau_1(H)$ takes values only in the interval of a small length $\ell \approx 0.110017$. Functions $\tau_k(H)$ take all values between their two limits: p_k at $H = 0$ and $\tau_k(H_{k*})$ at $H = H_{k*}$. For the knot $K_{3,4}$ in Fig. 8 we have $\tau_1(H_1) = 3/4$ (see Fig. 6) and for the knot $K_{2,3}$ we get $\tau_k(h_k) = 2/3$, see Fig. 7. The limit values at $k \longrightarrow \infty$ are $p_k \longrightarrow 1/2$, $\tau_k(H_{k*}) \longrightarrow 1/\sqrt{2} \approx 0.7071$, $|H_{k*}| \longrightarrow 1$.

Proposition 3. *Exact vector fields $\mathbf{V}_{\tilde{f}}(r,z)$ (2.21) defined by the eigenfunctions $\tilde{f}(r,z)$ (2.18) satisfying conditions*

$$a_1 + \cdots + a_n = 1, \quad a_k \geq 0, \quad |z_k| < \varepsilon << 1, \tag{7.4}$$

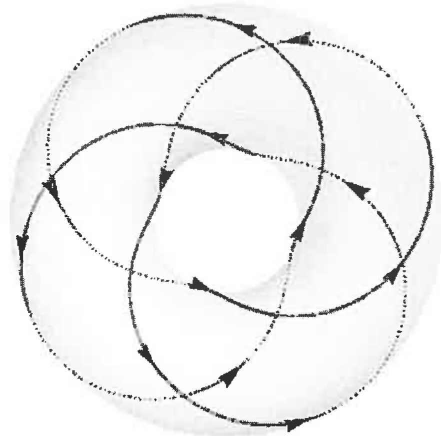

Fig. 8. The torus knot $K_{3,4}$ with period $\tau_1(H_1) = 3/4$ is realized by closed vortex lines in the first invariant ball $\mathbb{B}^3_{a_1}$.

for a sufficiently small ε provide a family of solutions to the Euler equations which possess a non-degenerate integrability of dynamics of vortex lines.

Proof. Any vector field $\mathbf{V}_f(r,z)$ (2.21) together with the pressure $p = C - \rho_c|\mathbf{V}_f(r,z)|^2/2$ is an exact solution to the Euler equations (1.3). The eigenfunctions $\tilde{f}(r,z)$ (2.18) defined by conditions (7.4) are ε-small perturbations of the eigenfunction $G_1(r,z)$ (2.17). The corresponding vector fields $\mathbf{V}_{\tilde{f}}(r,z)$ (2.21) define integrable dynamical systems (4.2)–(4.3) that are ε-small perturbations of the integrable system (6.3)–(6.4) for the eigenfunction $G_1(r,z)$ (2.17). Since the period $\tau(H)$ of closed trajectories C_H depends continuously on the small perturbations, we obtain that function of periods $\tilde{t}(\tilde{H})$ for the perturbed vector fields $\mathbf{V}_{\tilde{f}}(r,z)$ also is not constant. Hence by Proposition 1 the integrability of the perturbed dynamical system (4.2)–(4.3) is non-degenerate. □

8. Moduli space $\mathcal{S}(\mathbb{R}^3)$ of vortex knots

8.1. Let us consider trajectories of dynamical system (4.2)–(4.3) on the invariant tori $\mathbb{T}^2_H = C_H \times S^1$ where closed curves $C_H \subset R^2$ are defined by equations $H(r,z) = H =$ const. and circle S^1 corresponds to the angular variable φ. Suppose that for a value $H = H_0$ the function of periods $\tau(H)$ has a rational value $\tau(H_0) = p/q$ with coprime p and q. Then all trajectories of system (4.2)–(4.3) on torus $\mathbb{T}^2_{H_0}$ make q complete turns over its meridians and p complete turns over the longitudes. Hence all these trajectories form a torus knot $K_{p,q}$. An important invariant of any knot $K \subset \mathbb{R}^3$ is its Alexander polynomial which is an invariant of the fundamental group $\pi_1(\mathbb{R}^3 - K)$ of the complement of the knot K. The Alexander polynomial for a torus knot $K_{p,q}$ has the form [11,17]:

$$\Delta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}. \tag{8.1}$$

The Alexander polynomial is defined up to an arbitrary factor $\pm t^{n_1}$. Evidently, the knots $K_{p,q}$ and $K_{q,p}$ have the same polynomial (8.1). The polynomial $\Delta_{p,q}(t)$ has degree $n = pq + 1 - p - q = (p-1)(q-1)$. Since p and q are coprime, the degree n is always even. For $p/q = 1/q$ and for $p/q = p/1$ the polynomial $\Delta_{p,q}(t) \equiv 1$ and the corresponding closed curves are unknots.

The minimal degree n of the Alexander polynomial of a torus knot is 2. From the above equation $2 = n = (p-1)(q-1)$ we get $\{p,q\} = \{2,3\}$. The corresponding Alexander polynomial (8.1) is $\Delta_{2,3}(t) = t^2 - t + 1$.

8.2. There is only one quartic Alexander polynomial of a torus knot $K_{p,q}$. It corresponds to the knot $K_{2,5}$ and has the form $\Delta_{2,5}(t) = t^4 - t^3 + t^2 - t + 1$.

There are only two Alexander polynomials of degree six for the torus knots $K_{p,q}$. They correspond to the knots $K_{3,4}$ and $K_{2,7}$ and have the form $\Delta_{3,4}(t) = t^6 - t^5 + t^3 - t + 1$, $\Delta_{2,7}(t) = t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$.

Remark 12. The Lemma 6 below was first presented in [17] as Theorem 2.2.2. Its proof is given in sections 6.1.17 and 12.2.15 of [17] and is based on "Kurosh subgroup theorem" and uses the notions of "non-slice and non-amphicheiral" knots. Neither of those is necessary to prove Lemma 6; therefore we present here for the readers' convenience the new and straightforward proof.

Lemma 6. *If two torus knots $K_{p,q}$ and $K_{\tilde{p},\tilde{q}}$ are equivalent then either $\tilde{p}/\tilde{q} = p/q$ or $\tilde{q}/\tilde{p} = p/q$.*

Proof. If the two knots are equivalent then their Alexander polynomials (8.1) after multiplication by factors $\pm t^{n_i}$ coincide:

$$\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = \frac{(t^{\tilde{p}\tilde{q}}-1)(t-1)}{(t^{\tilde{p}}-1)(t^{\tilde{q}}-1)}. \tag{8.2}$$

Polynomial $\Delta_{p,q}(t)$ (8.1) does not have any real roots and all its complex roots lie on the unit circle $|t| = 1$. The root with minimal argument is $\tau_1 = \exp(2\pi i/(pq))$. The equality of polynomials (8.2) yields

$$\exp(2\pi i/(pq)) = \exp(2\pi i/(\tilde{p}\tilde{q})).$$

Hence we get $pq = \tilde{p}\tilde{q}$. Since the degrees of polynomials (8.2) coincide we have $pq+1-p-q = \tilde{p}\tilde{q}+1-\tilde{p}-\tilde{q}$. Therefore $p+q = \tilde{p}+\tilde{q}$. The two equalities $pq = \tilde{p}\tilde{q}$ and $p+q = \tilde{p}+\tilde{q}$ imply that either $\tilde{p} = p$, $\tilde{q} = q$ or $\tilde{q} = p$, $\tilde{p} = q$. □

Lemma 6 evidently implies the following

Corollary 4. *The torus knots $K_{p,q}$ and $K_{\tilde{p},\tilde{q}}$ with $p/q < 1$ and $\tilde{p}/\tilde{q} < 1$ are not equivalent if $p/q \neq \tilde{p}/\tilde{q}$.*

Remark 13. The torus knots $K_{3,4}$ and $K_{4,5}$ are realized by the vortex lines in the first invariant ball $\mathbb{B}^3_{a_1}$ (that corresponds to the invariant domain $\mathcal{D}_1 \times S^1$) because the fractions 3/4 and 4/5 are in the range of function $\tau_1(H)$, see formulae (7.3) and Fig. 6. Formulae (7.3) and Fig. 7 show that fractions 3/4 and 4/5 do not belong to the ranges of functions $\tau_k(H)$. Therefore applying Corollary 4 we find that the torus knots $K_{3,4}$ and $K_{4,5}$ are not realized in the invariant spherical shells between two invariant spheres $\mathbb{S}^2_{k-1}$ and $\mathbb{S}^2_k$ for any $k \geq 2$ (the shells correspond to the invariant domains $\mathcal{D}_k \times S^1$). The torus knot $K_{3,4}$ is shown in Fig. 8.

Theorem 1. *For the spheromak fluid flow (7.1) the moduli space $\mathcal{S}(\mathbb{R}^3)$ of vortex knots is naturally isomorphic to the set of all rational numbers p/q in the interval*

$$I_1: \quad \frac{1}{2} < \tau < M_1 = \frac{r_{1*}}{\sqrt{2(r_{1*}^2 - 2)}} = \tau_1(H_{1*}) \approx 0.8252, \tag{8.3}$$

where $r_{1*} \approx 2.7437$ is the first positive solution (5.4) to the equation $\tan r = r/(1-r^2)$. The torus knots $K_{p,q}$ with $p/q \in I_1$ are mutually non-equivalent. All vortex torus knots $K_{p,q}$ have a clockwise rotation around the meridians.

Proof. Proposition 1 implies that for all fractions p/q from the range of a function of periods $\tau_k(H)$ the torus knots $K_{p,q}$ are realized by the vortex lines in the invariant domain $\mathcal{D}_k \times S^1$, $k \geq 1$. For $k = 1$ the domain $\mathcal{D}_1 \times S^1$ corresponds to the first invariant ball $\mathbb{B}^3_{a_1}$ and for $k \geq 2$ the domain $\mathcal{D}_k \times S^1$ corresponds to the invariant spherical shell between two invariant spheres $\mathbb{S}^2_{k-1}$ and $\mathbb{S}^2_k$. In Sections 5 and 6, we have shown that the lower and upper limits of ranges (7.2) of functions $\tau_k(H)$ monotonously decrease when $k \longrightarrow \infty$:

$$p_k \downarrow \frac{1}{2}, \quad \tau_k(H_{k*}) = \frac{r_{k*}}{\sqrt{2(r^2_{k*} - 2)}} \downarrow \frac{1}{\sqrt{2}}. \tag{8.4}$$

Hence the union of the ranges (7.2) of all functions $\tau_k(H)$ is the interval I_1 (8.3). Hence for any fraction $p/q \in I_1$ the corresponding torus knot $K_{p,q}$ is realized by the closed vortex lines.

The mutual non-equivalence of the torus knots $K_{p,q}$ follows from Corollary 4 since $p/q < \tau_1(H_{1*}) \approx 0.8252 < 1$. Their clockwise rotation around the meridians follows from Corollary 2 of Section 4. □

Let us define two intervals I_3 and I_4:

$$I_3 : \quad \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right), \qquad I_4 : \quad \left(\frac{1}{\sqrt{2}}, M_1 = \frac{r_{1*}}{\sqrt{2(r^2_{1*} - 2)}}\right). \tag{8.5}$$

Proposition 4.

(a) Any torus knot $K_{p,q}$ with p/q in the interval I_3 (8.5) is realized by the vortex lines in infinitely many invariant domains $\mathcal{D}_k \times S^1$.

(b) Any torus knot $K_{\tilde{p},\tilde{q}}$ with $\tilde{p}/\tilde{q}$ in the interval I_4 (8.5) is realized only in finitely many invariant domains $\mathcal{D}_k \times S^1$.

Proof. *(a)* Using the limits (8.4), we see that any fraction $p/q \in I_3$ belongs to the ranges of all functions $\tau_k(H)$ starting from some k_1. Therefore the corresponding knot $K_{p,q}$ is realized in all invariant domains $\mathcal{D}_k \times S^1$ for $k \geq k_1$.

(b) Since $\tilde{p}/\tilde{q} > 1/\sqrt{2}$ and $\lim_{k\longrightarrow\infty} \tau_k(H_{k*}) = 1/\sqrt{2}$ we find from (7.2) that the $\tilde{p}/\tilde{q} \in I_4$ does not belong to the ranges of all functions $\tau_k(H)$ starting from some integer k_2. Hence applying Corollary 4 we see that the torus knot $K_{\tilde{p},\tilde{q}}$ is realized by the vortex lines only in finitely many invariant domains $\mathcal{D}_k \times S^1$ for $1 \leq k < k_2$. □

Remark 14. For $k \geq 2$, the ranges of all functions $\tau_k(H)$ for domains $\mathcal{D}_k$ contain the fraction $2/3 = \tau_k(h_k)$, see formulae (7.3) and Fig. 7. Hence the torus knot $K_{2,3}$ (trefoil knot) is realized by vortex lines in all invariant domains $\mathcal{D}_k \times S^1$ for $k \geq 2$ at $H = h_k$. However it is not realized as a vortex knot in domain $\mathcal{D}_1 \times S^1$ because $2/3$ is not in the range of function $\tau_1(H)$, see Fig. 6.

9. Moduli spaces $\mathcal{S}_m(\mathbb{B}^3_a)$ of vortex knots

9.1. In this Section we find the moduli spaces of vortex knots for the axisymmetric fluid flows $\mathbf{V}(\mathbf{x})$ inside a ball $\mathbb{B}^3_a$ $(x^2 + y^2 + z^2 \leq a^2)$ which are solutions to the equations

$$\operatorname{curl} \mathbf{V}(\mathbf{x}) = \lambda \mathbf{V}(\mathbf{x}), \quad (\mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})|_{\partial \mathbb{B}^3_a} = 0. \tag{9.1}$$

Here $\mathbf{n}(\mathbf{x}) = \mathbf{x}/a$ is the unit normal vector field on the boundary sphere $\mathbb{S}^2_a = \partial\mathbb{B}^3_a$. The second equation in (9.1) evidently is the non-penetration condition and means that the boundary sphere $\mathbb{S}^2_a$ is invariant under the flow $\mathbf{V}(\mathbf{x})$. The equations (9.1) mean that the vector field $\mathbf{V}(\mathbf{x})$ is a solution to the boundary eigenvalue problem [8,19,24,34,38] for the curl operator on the ball $\mathbb{B}^3_a$.

Let us show in a straightforward way that the spheromak vector field $\mathbf{V}_1(x, y, z)$ (7.1) generates an infinite series of exact axisymmetric solutions to the boundary eigenvalue problem (9.1):

$$\mathbf{V}_{1m}(x, y, z) = \mathbf{V}_1(\lambda_m x, \lambda_m y, \lambda_m z), \quad \lambda_m = a^{-1} R_m, \tag{9.2}$$

where R_m is the m-th positive solution to the equation $\tan R = R$ (3.6). Since the spheromak vector field $\mathbf{V}_1(x, y, z)$ (2.14), (7.1) satisfies equation (2.1), we find that for any λ vector field $\mathbf{V}_1(\lambda x, \lambda y, \lambda z)$ satisfies Beltrami equation $\operatorname{curl} \mathbf{V}(\mathbf{x}) = \lambda \mathbf{V}(\mathbf{x})$. The first integral $H(r, z) = -r^2 G_2(R)$ (3.1) of the vector field $\mathbf{V}_1(x, y, z)$ (2.14) yields the first integral $H_\lambda(r, z) = -\lambda^2 r^2 G_2(\lambda R)$ of the vector field $\mathbf{V}_1(\lambda x, \lambda y, \lambda z)$. Since all invariant submanifolds $H(r, z) = C$ with a non-zero constant $C \neq 0$ are tori $\mathbb{T}^2_C$, we get that the sphere $\mathbb{S}^2_a$ of radius a is an invariant submanifold for the vector field $\mathbf{V}_1(\lambda x, \lambda y, \lambda z)$ if and only if the equation $H_\lambda(r, z) = -\lambda^2 r^2 G_2(\lambda a) = 0$ holds on $\mathbb{S}^2_a$. Therefore λa must satisfy the equation $G_2(R) = 0$ (3.2) that implies that λa must be equal to one of the roots R_m of equation $\tan R = R$ (3.6). Hence we get an infinite series of eigenvalues $\lambda_m = R_m/a$ and eigenvector fields $\mathbf{V}_{1m}(x, y, z)$ (9.2) for the boundary eigenvalue problem (9.1).

Theorem 2. *The moduli space $\mathcal{S}_m(\mathbb{B}^3_a)$ of vortex knots for the fluid flow $\mathbf{V}_{1m}(x, y, z)$ (9.2) inside a ball $\mathbb{B}^3_a$ is naturally isomorphic to the set of all rational numbers p/q in the interval*

$$J_m : \quad \frac{1}{2\pi}(R_m - R_{m-1}) < \tau < M_1 \approx 0.8252, \tag{9.3}$$

where number M_1 is defined by equation (8.3). The space $\mathcal{S}_m(\mathbb{B}^3_a)$ does not depend on the radius a. The torus knots $K_{p,q}$ with $p/q \in J_m$ are mutually non-equivalent. All vortex torus knots $K_{p,q}$ have a clockwise rotation around the meridians.

Proof. The equation for the vortex lines for the m-th flow (9.2) has the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \operatorname{curl} \mathbf{V}_1(\lambda_m \mathbf{x}). \tag{9.4}$$

Substituting here Beltrami equation (9.1) and multiplying with λ_m we get

$$\frac{\mathrm{d}(\lambda_m \mathbf{x})}{\mathrm{d}t} = \lambda_m^2 \mathbf{V}_1(\lambda_m \mathbf{x}).$$

Hence the vortex lines for the vector field (9.2) inside the ball $\mathbb{B}^3_a$ ($|\mathbf{x}| \leq a$) after change of time $\mathrm{d}\tau/\mathrm{d}t = \lambda_m^2$ and substitution $\lambda_m \mathbf{x} = \mathbf{y}$ satisfy equation

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \operatorname{curl} \mathbf{V}_1(\mathbf{y}) = \mathbf{V}_1(\mathbf{y}). \tag{9.5}$$

Since $|\mathbf{y}| = |\lambda_m||\mathbf{x}| \leq R_m$, the vortex lines (9.4) for the flow (9.2) inside the ball $\mathbb{B}^3_a$ are mapped by the diffeomorphism $\mathbf{y} = \lambda_m \mathbf{x}$ into the vortex lines (1.12), (9.5) for the spheromak fluid flow $\mathbf{V}_1(x, y, z)$ (7.1) inside the invariant sphere $\mathbb{S}^2_m$ of radius R_m.

The interior of the invariant with respect to the flow (9.5) sphere $\mathbb{S}^2_m$ is the union of m invariant domains $\mathcal{D}_k \times S^1$, $k = 1, 2, \cdots, m$, the interval $-R_m < z < R_m$, $r = 0$ and $m - 1$ intermediate invariant spheres $\mathbb{S}^2_k$, $k = 1, 2, \cdots, m-1$. In each domain $\mathcal{D}_k \times S^1$, the function of periods $\tau_k(H)$ (see Section 4) is changing in the interval (7.2):

$$p_k = \frac{1}{2\pi}(R_k - R_{k-1}) < \tau_k(H) < \tau_k(H_{k*}).$$

Since any two subsequent intervals $(p_k, \tau_k(H_{k*}))$ and $(p_{k+1}, \tau_{k+1}(H_{k+1*}))$ have non-zero intersection and both bounds p_k and $\tau_k(H_{k*})$ monotonously decrees when k grows we find that the union of intervals $(p_k, \tau_k(H_{k*}))$ for $k = 1, 2, \cdots, m$ is the interval J_m (9.3).

As is shown in Section 4, any vortex torus knot $K_{p,q}$ of the spheromak flow in $\mathbb{R}^3$ corresponds to the rational value p/q of some function of periods $\tau_k(H)$ and vice versa. All realized by the system (9.4), (9.5) vortex knots $K_{p,q}$ have a clockwise rotation around the meridians because by virtue of Theorem 1 this is true for all vortex knots for the spheromak flow (7.1). Hence using Theorem 1 we obtain that the moduli space $\mathcal{S}_m(\mathbb{B}_a^3)$ of vortex knots for the m-th flow (9.2) is isomorphic the set of all rational numbers p/q in the interval J_m (9.3) and does not depend on the radius a of the ball $\mathbb{B}_a^3$. □

Using formulae (6.12) for the numbers $p_k = (R_k - R_{k-1})/(2\pi)$ for $k = 1, 2, 3, 4$ we find the approximate bounds of the first four intervals J_m (9.3):

$$J_1: \quad (0.7151,\ 0.8252), \quad J_2: \quad (0.5144,\ 0.8252),$$
$$J_3: \quad (0.5062,\ 0.8252), \quad J_4: \quad (0.5033,\ 0.8252).$$

In the limit $m \longrightarrow \infty$ we find from equation (6.13) that $J_m \longrightarrow I_1$ where the interval $I_1:\ (1/2,\ 0.8252)$ is defined by formulae (8.3).

10. Conclusion

We have derived exact solutions to the steady Euler equations (1.4) with velocity vector fields

$$\mathbf{V}_f(\mathbf{x}) = \frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial r}\right)\hat{\mathbf{e}}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\hat{\mathbf{e}}_z - \frac{\partial f}{\partial r}\hat{\mathbf{e}}_\varphi, \tag{10.1}$$

and pressure $p = C - \rho_c|\mathbf{V}_f(\mathbf{x})|/2$, where functions $f(r,z)$ have the form

$$f_N = \sum_{k=1}^{N} a_k \frac{\sin\sqrt{r^2 + (z - z_k)^2}}{\sqrt{r^2 + (z - z_k)^2}}, \quad G_1 = \frac{\sin\sqrt{r^2 + z^2}}{\sqrt{r^2 + z^2}}, \tag{10.2}$$

where a_k, z_k are arbitrary constants. The functions $f(r,z)$ are eigenfunctions of the Laplace operator: $\Delta f = -f$ (2.16). The exact solutions (10.1)–(10.2) satisfy also the Beltrami equation $\operatorname{curl}\mathbf{V} = \mathbf{V}$ (2.1).

We have proved that dynamical systems of vortex lines $\mathrm{d}\mathbf{x}/\mathrm{d}t = \operatorname{curl}\mathbf{V}_f(\mathbf{x})$ are integrable and their dynamics occurs on invariant tori $\mathbb{T}_H^2 = C_H \times S^1$, defined by equation $H(\mathbf{x}) = H = \text{const.}$, $H(\mathbf{x}) = -r\partial f(r,z)/\partial r$. Here $C_H \subset R^2$ is a closed curve $H(r,z) = H = \text{const.}$ and circle S^1 corresponds to the angular variable φ. For function $f = G_1$, system (1.12) has infinitely many invariant domains $\mathcal{D}_k \times S^1$ bounded by the spheres $R = R_k$ and $R = R_{k-1}$ satisfying equation $H(\mathbf{x}) = 0$; the numbers R_k are roots of the equation $\tan R = R$.

The rational values of functions of periods $\tau_k(H) = p/q$ define tori $\mathbb{T}_H^2$ on which all trajectories of system (1.12) are closed curves and make q complete turns around the meridians and p complete turns around the longitudes. These trajectories form the torus knots $K_{p,q}$ (p and q are coprime).

For the exact spheromak fluid flow $\mathbf{V}_s(\mathbf{x}) = \mathbf{V}_{G_1}(\mathbf{x})$ (7.1), (10.1), we have demonstrated in Theorem 1 that the moduli space $\mathcal{S}(\mathbb{R}^3)$ of all non-isotopic vortex knots in $\mathbb{R}^3$ is naturally isomorphic to the set of all rational numbers p/q in the interval $I_1:\ 1/2 < \tau < M_1 \approx 0.8252$. In Proposition 4 we proved that torus knots $K_{p,q}$ with $1/2 < p/q < 1/\sqrt{2}$ are realized on infinitely many invariant tori $\mathbb{T}_H^2 \subset \mathcal{D}_k \times S^1$ for $k \geq 2$, while torus knots with $1/\sqrt{2} < p/q < M_1$ are realized only on finitely many tori.

For the spheromak flow $\mathbf{V}_s(\mathbf{x})$ (7.1) in $\mathbb{R}^3$ we proved in Theorem 1 that all its vortex knots are torus knots $K_{p,q}$ with rational numbers p/q belonging to the short interval $(0.5, 0.8252)$ and not with any rationals from the infinite interval $(0, \infty)$. This gives a counterexample to Moffatt's statements of [21–23] that for the spheromak fluid flow $\mathbf{V}_s$ (that is one of the flows studied in [21], pp. 126–129) and for the spheromak magnetic field $\mathbf{B}_s$ [22], pp. 30–31, all torus knots $K_{p,q}$ for any rational numbers $p/q \in (0,\infty)$ are realized as vortex knots (correspondently as the magnetic field $\mathbf{B}_s$ knots).

In Section 9 we have shown that the spheromak fluid flows $\mathbf{V}_{1m}(x,y,z) = \mathbf{V}_{G_1}(\lambda_m x, \lambda_m y, \lambda_m z)$ are solutions to the boundary eigenvalue problem for the curl operator on a ball $\mathbb{B}_a^3$ of radius a, provided that $\lambda_m = R_m/a$ and $\tan R_m = R_m$. We have proved in Theorem 2 that the corresponding moduli space $\mathcal{S}_m(\mathbb{B}_a^3)$ of vortex knots is naturally isomorphic to the set of all rational numbers in the interval J_m : $(R_m - R_{m-1})/(2\pi) < \tau < M_1 \approx 0.8252$, where R_k is the k-th positive root of equation $\tan R = R$. Therefore the moduli spaces $\mathcal{S}_m(\mathbb{B}_a^3)$ do not depend on the radius a of the ball $\mathbb{B}_a^3$, all spaces $\mathcal{S}_m(\mathbb{B}_a^3)$ are different (for different m's) and approximate the moduli space $\mathcal{S}(\mathbb{R}^3)$ because $(R_m - R_{m-1})/(2\pi) \longrightarrow 1/2$ when $m \longrightarrow \infty$.

In view of the equivalence of equations (1.4) and (1.5) as well as equations (1.4) and (1.7) the above results are equally applicable to the moduli spaces of knots formed by the magnetic field lines for the solutions $\mathbf{B}_f(\mathbf{x})$ and $\mathbf{B}_{G_1}(\mathbf{x})$ (10.1)–(10.2) to the plasma equilibrium equations (1.5) with pressure $\tilde{p} = \text{const.}$ and to the more general MHD equilibria (1.7) with $\mathbf{V}(\mathbf{x}) = \gamma \mathbf{B}(\mathbf{x})$.

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