Rapid Communication

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Invariants of the Axisymmetric Plasma Flows

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Abstract: Infinite families of new functionally independent invariants are derived for the axisymmetric dynamics of viscous plasmas with zero electrical resistance. As a consequence, we find that, if two axisymmetric plasma states are dynamically connected, then their total number of magnetic rings must be equal (the same as for the total numbers of magnetic blobs) and the corresponding infinitely many new invariants must coincide.

Keywords: Invariants; Magnetic Blobs; Magnetic Helicity; Magnetic Rings; Safety Factor.

1 Introduction

As is known [1, 2], the magnetohydrodynamic (MHD) equations with isotropic viscosity and zero electrical resistance have the form

$$\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla p+\frac{1}{\rho\mu}(\nabla\times\mathbf{B})\times\mathbf{B}+\nu\Delta\mathbf{V}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{B}), \tag{2}$$

$$\nabla\cdot\mathbf{V}=0,\ \ \nabla\cdot\mathbf{B}=0,\ \ \frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho=0. \tag{3}$$

For incompressible plasma flows with velocity $\mathbf{V}(\mathbf{x}, t)$, (2) implies that the magnetic field $\mathbf{B}(\mathbf{x}, t)$ is transported in time with the plasma flow (or "is frozen in the flow"). Here, $\rho(\mathbf{x}, t)$ is the plasma density, which according to the last equation of (3) is preserved along the plasma streaklines, μ is the magnetic permeability, $p(\mathbf{x}, t)$ is the pressure, ν is the kinematic viscosity, ∇ is the nabla operator, and Δ is the Laplace operator. We assume that during some finite time t, the vector functions $\mathbf{V}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$, and the scalar functions $\rho(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are of class C^2. The special regimes of plasma relaxation are derived in the explicit form in [3]. Symmetry transforms for the ideal MHD equilibria are derived in [4]. Exact axisymmetric plasma equilibria are presented in [5] and the helically symmetric ones in [6].

The concrete form $\nu\Delta\mathbf{V}$ of the dissipative term in (1) can be altered for a more sophisticated formula, for example, for the one recently introduced in [7]. However, this would not have changed our results because they are based entirely on the MHD equations (2) and (3). Therefore, we use in (1) the classical viscosity term $\nu\Delta\mathbf{V}$ as in the Navier-Stokes equations.

As is known [1, 2], the frozenness of magnetic field $\mathbf{B}$ into the plasma flow leads to such invariants as the integer-valued Gauss linking number for any two closed magnetic field lines [8] and the discrete topological invariants of magnetic field knots [9].

Woltjer's integral H_D [10] of the magnetic helicity has the form

$$H_D=\int_D \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3x, \tag{4}$$

where $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$.

Woltjer's magnetic helicity $H_D=\int_D\mathbf{A}\cdot[\nabla\times\mathbf{A}]\mathrm{d}^3x$ [10, 11] has an analogue, which is the hydrodynamic helicity $\mathcal{H}_D=\int_D\mathbf{V}\cdot[\nabla\times\mathbf{V}]\mathrm{d}^3x$. The well-known distinction between the two helicities is that the magnetic helicity H_D (4) is not uniquely defined by the magnetic field $\mathbf{B}$ because H_D depends on the gauge transformations of the vector potential $\mathbf{A}$. In contradistinction to this, the hydrodynamic helicity $\mathcal{H}_D$ is uniquely defined by the velocity field $\mathbf{V}$ because there are no gauge transformations of the velocity $\mathbf{V}$.

The hydrodynamic helicity $\mathcal{H}_D$ satisfies the equation

$$\frac{\mathrm{d}\mathcal{H}_D}{\mathrm{d}t}=\oint_S\left[-F(\rho)+\frac{1}{2}|\mathbf{V}|^2\right][\nabla\times\mathbf{V}]\cdot\mathbf{n}\,\mathrm{d}s, \tag{5}$$

where the surface $S=\partial D$ is the boundary of the domain $D\subset\mathbb{R}^3$, vector field $\mathbf{n}(\mathbf{x})$ is the outward unit vector field orthogonal to S, and ds is the area element of surface S. Equation (5) was first derived by Moffatt in [8], where it was assumed that the pressure p is a function of the density ρ only (the barotropic condition $p=p(\rho)$) and $\nabla F(\rho)=\rho^{-1}\nabla p$. In ([12], p. 108) Serre has shown that, in general (if $[\nabla\times\mathbf{V}]\cdot\mathbf{n}\neq 0$), the helicity $\mathcal{H}_D$ inside an invariant domain D (with the non-penetration condition $\mathbf{V}\cdot\mathbf{n}=0$ on S) is not constant in time.

Helicity was studied by Kudryavtseva [13, 14] and by Enciso et al. [15]. The authors stated that for divergence-free

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vector fields the integral invariants (i.e. the integrals of certain densities "over compact three-dimensional manifold M without boundary" ([15], p. 2035) are often related to helicities.

We derive in this paper new integral invariants of the axisymmetric plasma flows with zero electrical resistance, which depend on the magnetic field **B** and in some cases on the plasma density ρ but do not depend on any gauge transformations of the vector potential **A**. Our key idea is to study the safety factor q [2, 16, 17] for the time-dependent axisymmetric magnetic fields **B** and to prove that the emerging function $q(\mathbf{x}, t)$ is conserved along the plasma streaklines.

In Section 2 we define the magnetic rings and magnetic blobs for the axisymmetric dynamics of plasma with zero electrical resistance and prove that the former are frozen into the plasma flows. In Section 3 we calculate the Woltjer's helicity H_D (4) for the axisymmetric magnetic rings and blobs and demonstrate how it is altered by the axisymmetric gauge transformations of the vector potential **A**. In Section 5 we prove that the safety factor $q(\mathbf{x}, t)$ (which is uniquely defined by the magnetic field **B** in the domains where all magnetic surfaces are tori or magnetic axes) is conserved along the plasma streaklines. In Sections 6 and 7 we prove that the axisymmetric plasma flows preserve infinitely many new invariants, which are either functional invariants or the integrals of the arbitrary C^2-functions of the safety factor $q(\mathbf{x}, t)$ over the magnetic rings and magnetic blobs, for example, $q^n(\mathbf{x}, t)$, $n=1, 2, \ldots$. The functional invariants are the values of certain axisymmetric functions on the magnetic axes of the axisymmetric field $\mathbf{B}(\mathbf{x}, t)$. In Section 7 we demonstrate that infinitely many new integral invariants with possibly one exception are functionally independent of the magnetic helicity.

2 Magnetic Rings and Blobs

Let $\mathbf{A}(r, z, t)$ be a z-axisymmetric vector potential

$$\mathbf{A}(r, z, t)=u(r, z, t)\hat{\mathbf{e}}_r+v(r, z, t)\hat{\mathbf{e}}_z+r^{-1}\psi(r, z, t)\hat{\mathbf{e}}_\varphi, \tag{6}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\varphi$ are vectors of unit length along the axes of the cylindrical coordinates $r\geq 0$, z, φ. The z-axisymmetric gauge transformation of the vector potential **A** is

$$\tilde{\mathbf{A}}(r, z, t)=(u+\partial f/\partial r)\hat{\mathbf{e}}_r+(v+\partial f/\partial z)\hat{\mathbf{e}}_z+r^{-1}(\psi+\chi(t))\hat{\mathbf{e}}_\varphi, \tag{7}$$

where u, v, f, ψ are C^2-functions of r, z, and t, and $\chi(t)$ is a C^1-function. The gauge transformation (7) is well defined in any (non-simply-connected) toroidal domain $T^3=D^2\times\mathbb{S}^1$ that lies in the space $r>0$. Here, D^2 is a compact domain in the plane $(r>0, z)$ and $\mathbb{S}^1$ is the circle corresponding to the anglular variable φ. The gauge transformation of function $\psi(r, z, t)$ is

$$\tilde{\psi}(r, z, t)=\psi(r, z, t)+\chi(t). \tag{8}$$

The magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ has the form

$$\mathbf{B}(r, z, t)=-r^{-1}\partial\psi(r, z, t)/\partial z\hat{\mathbf{e}}_r+r^{-1}\partial\psi(r, z, t)/\partial r\hat{\mathbf{e}}_z+w(r, z, t)\hat{\mathbf{e}}_\varphi, \tag{9}$$

and is evidently gauge-invariant:

$$\tilde{\mathbf{B}}(r, z, t)=\mathbf{B}(r, z, t),\quad \tilde{w}(r, z, t)=w(r, z, t)=\partial u/\partial z-\partial v/\partial r. \tag{10}$$

The gauge transformation (8) implies that the flux function $\psi(r, z, t)$ is not uniquely defined by the magnetic field $\mathbf{B}(r, z, t)$ (9).

Equation (9) yields that the derivative of the flux function $\psi(r, z, t)$ along the plasma streaklines is

$$\frac{D\psi}{Dt}=\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial r}V_r+\frac{\partial\psi}{\partial z}V_z=\frac{\partial\psi}{\partial t}+r(B_zV_r-B_rV_z). \tag{11}$$

Here, the initial poloidal components B_r, B_z of the magnetic field $\mathbf{B}(r, z, t)$ and V_r, V_z for the plasma velocity $\mathbf{V}(r, z, t)$ can be choosen independently. For unsteady solutions, (11) yields that the generic flux function $\tilde{\psi}(r, z, t)$ is not conserved along the plasma streaklines. Indeed, assuming that a special flux function $\psi(r, z, t)$ is conserved ($D\psi/Dt=0$), we get that all functions $\tilde{\psi}(r, z, t)$ (8) with $\mathrm{d}\chi(t)/\mathrm{d}t\neq 0$ are not conserved because

$$\frac{D\tilde{\psi}}{Dt}=\frac{D\psi}{Dt}+\frac{\mathrm{d}\chi(t)}{\mathrm{d}t}\neq 0. \tag{12}$$

For any constant time t, the magnetic field $\mathbf{B}(r, z, t)$ (9) is tangent to the z-axisymmetric magnetic surfaces $\psi(r, z, t)=C(t)$. The family of magnetic surfaces for the arbitrary function $C(t)$ is invariant under the gauge transformations (8).

Dynamics along magnetic field lines for $t=\text{const}$ is defined by the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}=\frac{\mathrm{d}x}{\mathrm{d}\tau}\hat{\mathbf{e}}_x+\frac{\mathrm{d}y}{\mathrm{d}\tau}\hat{\mathbf{e}}_y+\frac{\mathrm{d}z}{\mathrm{d}\tau}\hat{\mathbf{e}}_z=\frac{\mathrm{d}r}{\mathrm{d}\tau}\hat{\mathbf{e}}_r+r\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\hat{\mathbf{e}}_\varphi+\frac{\mathrm{d}z}{\mathrm{d}\tau}\hat{\mathbf{e}}_z=\mathbf{B}(r, z, t),$$

which by virtue of (9) has the form

$$\frac{\mathrm{d}r}{\mathrm{d}\tau}=-r^{-1}\frac{\partial\psi(r, z, t)}{\partial z},\quad \frac{\mathrm{d}z}{\mathrm{d}\tau}=r^{-1}\frac{\partial\psi(r, z, t)}{\partial r}, \tag{13}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}=r^{-1}w(r, z, t). \tag{14}$$

The system (13), (14) is evidently invariant under the gauge transformations (8), (10). After the time change $d\tau_1/d\tau = r^{-1}$, the system (13) for $t = \text{const}$ becomes a τ_1-autonomous Hamiltonian system.

Hamiltonian systems were applied to the “plasma-confining magnetic fields” by Kerst [18], see also pages 1062–1063 of review [19]. We derive the Hamiltonian system (13) in the poloidal coordinates (r, z) and not in the Cartesian ones (x, y) as in [18, 19]. The system (13) is derived above straight from the definition $\mathbf{B} = \nabla \times \mathbf{A}$ and without any assumption of “a uniform B_z” as in ([18], p. 253).

Boozer introduced in [20] for the Hamiltonian treatment of the plasma equilibria “generalised magnetic coordinates”, see also the review ([21], p. 1076). We do not use them because the standard cylindrical ones (r, z, φ) are sufficient for our study.

We will use the following classical properties of any autonomous Hamiltonian system in $\mathbb{R}^2$ [22]:

(α) Its Hamiltonian function is a first integral (for system (13) it is the flux function $\psi(r, z, t)$ with a fixed time t);
(β) All its non-degenerate equilibria are either saddles s_ℓ or centers c_k;
(γ) It preserves the area.

These properties imply that every closed trajectory $C_{\psi(t)}$ ($\psi(r, z, t) = \text{const}$) of system (13) for a fixed time t belongs to one of the two-dimensional sets $D_k(t)$ in the poloidal plane (r, z) satisfying the following conditions:

1. $D_k(t)$ is invariant with respect to the system (13);
2. the set $D_k(t)$ is connected and closed;
3. all trajectories of system (13) in the set $D_k(t)$ have finite Euclidean length and are either closed curves or separatrices of the equilibrium points in $D_k(t)$;
4. a dense open subset of $D_k(t)$ is filled with closed trajectories of system (13);
5. the set $D_k(t)$ is the largest among the sets satisfying conditions 1–4 and having a non-empty intersection with $D_k(t)$.

We will call each set $D_k(t)$ satisfying conditions 1–5 a maximal set (the term “maximal” refers to the condition 5). Figure 1 shows the phase portrait of a concrete system (13) that has eight maximal sets $D_1, \ldots, D_8$.

The boundary $\partial D_k(t)$ is a one-dimensional set $\mathcal{C}_k(t)$ that is invariant with respect to the dynamical system (13). Therefore, $\mathcal{C}_k(t)$ satisfies the equation $\psi(r, z, t) = b_k(t)$. System (13) preserves the poloidal area $dS = 2\pi r\, dr\, dz$.

The closed trajectories $C_{\psi(t)}$ of system (13) (for $t = \text{const}$) after rotation around the axis of symmetry z define the two-dimensional tori $\mathbb{T}^2_{\psi(t)} = C_{\psi(t)} \times \mathbb{S}^1 \subset \mathbb{R}^3$, which are invariant with respect to the system (13), (14) and therefore are called $\mathbf{B}(r, z, t)$-invariant. Here, the circle $\mathbb{S}^1$ corresponds to the angular variable φ. The rotation of a maximal set $D_k(t)$ around the z-axis defines a three-dimensional set $\mathcal{D}^3_k(t) = D_k(t) \times \mathbb{S}^1 \subset \mathbb{R}^3$, which is closed and $\mathbf{B}(r, z, t)$-invariant. Its 3D volume is

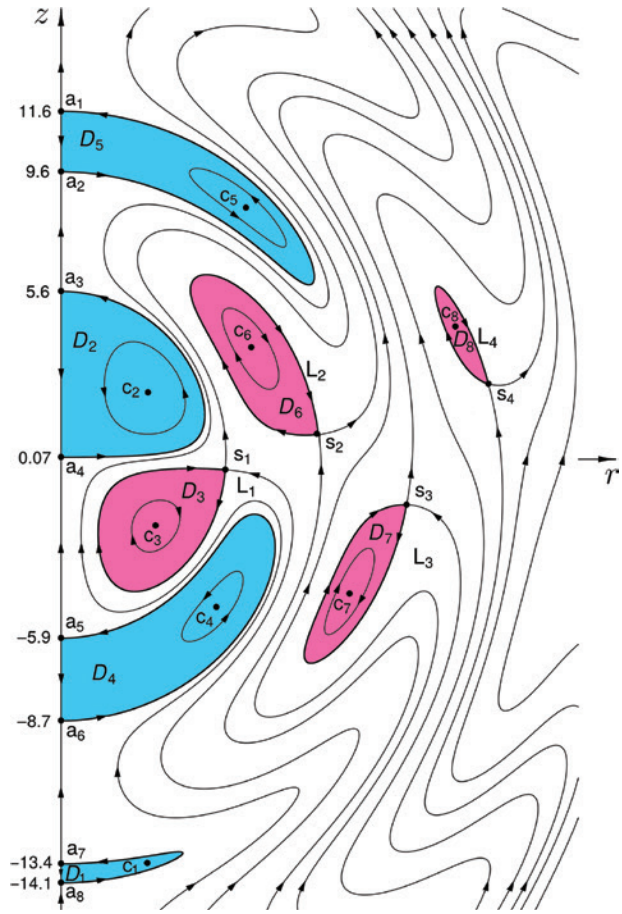

Figure 1: Poloidal sections of magnetic rings D_3, D_6, D_7, D_8 (pink) and magnetic blobs D_1, D_2, D_4, D_5 (blue) for the exact plasma equilibrium with the flux function $\psi(r, z) = r^2[0.005 - zR^{-4}((3 - R^2)R^{-1}\sin R - 3\cos R)]$, where $R = \sqrt{r^2 + z^2}$. Rotation of the sets D_3, D_6, D_7, D_8 around the axis z defines four magnetic rings and rotation of the sets D_1, D_2, D_4, D_5 around the axis z defines four magnetic blobs.

$$\mathrm{Vol}\,\mathcal{D}^3_k(t) = \int_{\mathcal{D}^3_k(t)} r\,dr\,dz\,d\varphi = 2\pi \int_{D_k(t)} r\,dr\,dz = \mathrm{Ar}_m D_k(t). \tag{15}$$

We call $\mathrm{Ar}_m D_k(t)$ the poloidal area of $D_k(t)$; it is equal to the 3D volume $\mathrm{Vol}\,\mathcal{D}^3_k(t)$. Rotation of the boundary curve $\mathcal{C}_k(t) = \partial D_k(t)$ around the axis z defines the boundary $\partial \mathcal{D}^3_k(t) = \mathcal{C}_k(t) \times \mathbb{S}^1$.

2.1 Magnetic Rings and Blobs

If $\mathcal{C}_k(t) = \partial D_k(t)$ is a closed curve lying in the domain $r > 0$, then the closed set $\mathcal{D}^3_k(t) = D_k(t) \times \mathbb{S}^1$ is a maximal

$\mathbf{B}(r, z, t)$-invariant ring $\mathcal{R}_k^3(t)$, which we call a *magnetic ring*. It is bounded by the torus $\mathcal{T}_k^2(t)=\mathcal{C}_k(t)\times\mathbb{S}^1$. Otherwise, a closed and bounded set $\mathcal{D}_k^3(t)$ is a blob $\mathcal{B}_k^3(t)$ (which we call a *magnetic blob*) containing a segment $S_k(t)$ of the axis of symmetry z $(r=0)$.

The gauge transformations (7), (8) are well defined in the magnetic rings $\mathcal{R}_k^3(t)$ since they lie in the domain $r>0$ and are not defined in the magnetic blobs $\mathcal{B}_k^3(t)$ because they contain the segments $S_k(t)$ of the axis $r=0$.

Example 1: Figure 1 shows the phase portrait of the dynamical system (13) for the exact plasma equilibrium which has the flux function $\psi(r,z)=r^2[0.005-zR^{-4}((3-R^2)R^{-1}\sin R-3\cos R)]$ and $w(r, z)=\psi(r, z)$, where $R=\sqrt{r^2+z^2}$. The plot in Figure 1 contains eight maximal sets D_k, which are densely filled with smooth, closed trajectories C_ψ $(\psi(r, z)=\text{const})$ of system (13) encircling the centre equilibrium points c_k, $k=1, \dots, 8$. The maximal sets D_1, D_2, D_4, and D_5 are bounded, respectively, by the pairs of separatrices of the saddle equilibria (a_7, a_8), (a_3, a_4), (a_5, a_6), and (a_1, a_2), correspondingly. The approximate values of the coordinate z for the points a_k are indicated in Figure 1. Rotation of the sets D_1, D_2, D_4, D_5 around the axis of symmetry z defines four magnetic blobs $\mathcal{B}_k^3$. The maximal sets D_3, D_6, D_7, D_8 are bounded by the loop separatrices L_1, L_2, L_3, L_4 of the saddle equilibria s_1, s_2, s_3, s_4. (By definition, a loop separatrix of a dynamical system begins and ends at the same saddle equilibrium point.) Rotation of the sets D_3, D_6, D_7, D_8 around the axis z defines four magnetic rings $\mathcal{R}_j^3$ which are bounded by the tori $\mathcal{T}_j^2=L_j\times\mathbb{S}^1$, $j=1, 2, 3, 4$.

3 Helicity in the Magnetic Rings and Blobs

Using formulas (6) and (9) and $\mathrm{d}v=r\mathrm{d}r\mathrm{d}z\mathrm{d}\varphi$, we find the following for Woltjer's helicity (4) in the magnetic ring or blob $\mathcal{D}_k^3(t)$:

$$H_k(t)=\int_{\mathcal{D}_k^3(t)}\mathbf{A}\cdot\mathbf{B}\mathrm{d}v=2\pi\int_{D_k(t)}[\psi_r v-\psi_z u+\psi(u_z-v_r)]\mathrm{d}r\mathrm{d}z$$
$$=2\pi\int_{D_k(t)}[(\psi v)_r-(\psi u)_z+2\psi w]\mathrm{d}r\mathrm{d}z. \quad (16)$$

Using Green's theorem and the equation $\psi(r, z, t)=b_k(t)$ on the boundary curve $\mathcal{C}_k(t)=\partial D_k(t)$, we get

$$\int_{D_k(t)}[(\psi v)_r-(\psi u)_z]\mathrm{d}r\mathrm{d}z=\oint_{\mathcal{C}_k(t)}\psi(vn_r-un_z)\mathrm{d}s$$
$$=b_k(t)\oint_{\mathcal{C}_k(t)}(vn_r-un_z)\mathrm{d}s,$$

where $\mathrm{d}s$ is the element of the arc length of the curve $\mathcal{C}_k(t)$, and the vector (n_r, n_z) is the outward unit normal vector to it. Applying Green's theorem again, we find

$$b_k(t)\oint_{\mathcal{C}_k(t)}(vn_r-un_z)\mathrm{d}s=b_k(t)\int_{D_k(t)}(v_r-u_z)\mathrm{d}r\mathrm{d}z$$
$$=-b_k(t)\int_{D_k(t)}w\mathrm{d}r\mathrm{d}z.$$

Hence we get

$$\int_{D_k(t)}[(\psi v)_r-(\psi u)_z]\mathrm{d}r\mathrm{d}z=-b_k(t)\int_{D_k(t)}w\mathrm{d}r\mathrm{d}z. \quad (17)$$

Substituting formula (17) into (16), we derive for the magnetic helicity

$$H_k(t)=\int_{\mathcal{D}_k^3(t)}\mathbf{A}\cdot\mathbf{B}\mathrm{d}v$$
$$=2\pi\int_{D_k(t)}[2\psi(r, z, t)-b_k(t)]w(r, z, t)\mathrm{d}r\mathrm{d}z. \quad (18)$$

As shown above, the gauge transformations (7), (8) are well defined in the magnetic rings $\mathcal{R}_k^3(t)$. Formulas (8) and (10) imply $\tilde{b}_k(t)=b_k(t)+\chi(t)$, $\tilde{w}(r, z, t)=w(r, z, t)$. Therefore, (18) yields for the magnetic helicity $\tilde{H}_k(t)$ after the gauge transformation (8), (10)

$$\tilde{H}_k(t)=H_k(t)+2\pi\chi(t)\int_{D_k(t)}w(r, z, t)\mathrm{d}r\mathrm{d}z. \quad (19)$$

Formula (19) demonstrates that the magnetic helicity $H_k(t)$ (16) is not invariant under the gauge transformations (8), (10) if $\int_{D_k(t)}w(r, z, t)\mathrm{d}r\mathrm{d}z\neq 0$. Indeed, the helicity $H_k(t)$ (16) becomes an arbitrary function $F_k(t)=\tilde{H}_k(t)$ after the gauge transformation (8) with $\chi(t)=[F_k(t)-H_k(t)]\left(2\pi\int_{D_k(t)}w(r, z, t)\mathrm{d}r\mathrm{d}z\right)^{-1}$. The helicity $\tilde{H}_k(t)$ becomes a constant C_k after the gauge transformation (8) with the gauge function

$$\chi_k(t)=[C_k-H_k(t)]\left(2\pi\int_{D_k(t)}w(r, z, t)\mathrm{d}r\mathrm{d}z\right)^{-1}. \quad (20)$$

4 Dynamics in the Poloidal Coordinates (*r, z*)

Suppose a z-axisymmetric magnetic field $\mathbf{B}(r, z, t_1)$ at $t=t_1$ has a closed magnetic field line L. Its dynamics in the poloidal coordinates (r, z) is described by system (13), which defines the corresponding closed trajectory $P(L)$ that satisfies the equation $\psi(r, z, t_1)=C$. The closed trajectory $P(L)$ in the plane (r, z) belongs to a maximal set $D(t_1)\subset\mathbb{R}^2$, which is densely filled with closed trajectories $C_{\psi(t_1)}$ of system (13), see Figure 1.

For the z-axisymmetric plasma flows, system (13) defines the so-called induced diffeomorphisms of the poloidal plane (r, z). The plasma flow by virtue of the equation $\operatorname{div}\mathbf{V}=0$ preserves the 3D volume $\mathrm{d}\Omega = r\mathrm{d}r\mathrm{d}z\mathrm{d}\varphi$. For the z-axisymmetric flows, we get that the induced diffeomorphisms of the poloidal plane (r, z) preserve the poloidal area $\mathrm{d}S=2\pi r\mathrm{d}r\mathrm{d}z$, the axis of symmetry z $(r=0)$, and the invariant domain $r>0$. The frozenness of the magnetic field lines into the plasma flow [1] implies for the z-axisymmetric case that the trajectories of system (13) are frozen into the induced dynamics of plasma in the poloidal coordinates (r, z). Therefore, the phase portrait of dynamical system (13) is changed in time t by the induced diffeomorphisms of the poloidal plane (r, z). Let $M_{t_1 t}$ be the induced diffeomorphism defined by the plasma dynamics from time t_1 to time $t>t_1$. Any closed trajectory $C_{\psi(t_1)}$ of system (13) in the invariant domain $r>0$ is transformed by the diffeomorphism $M_{t_1 t}$ into another closed trajectory $C_{\psi(t)}$ of system (13) in the same domain $r>0$. Since any maximal set $D_k(t_1)$ is densely filled with the closed trajectories $C_{\psi(t_1)}$, it is transformed by the diffeomorphism $M_{t_1 t}$ into the maximal set $D_k(t)$. Using the frozenness of the magnetic field $\mathbf{B}(r, z, t)$ into the plasma flow, it is easy to prove that the image $M_{t_1 t}(D_k(t_1))$ coincides with the maximal set $D_k(t)$.

Because of the conservation of the area $\mathrm{d}S=2\pi r\mathrm{d}r\mathrm{d}z$ by system (13), we get the equality of the poloidal areas $\mathrm{Ar}_m D_k(t_1)=\mathrm{Ar}_m D_k(t)=2\pi\int_{D_k(t)} r\mathrm{d}r\mathrm{d}z$. Since the induced diffeomorphisms $M_{t_1 t}$ preserve the axis of symmetry z $(r=0)$ and the domain $r>0$, we get that the total number N_r of magnetic rings $\mathcal{R}_k^3(t)=D_k(t)\times\mathbb{S}^1$ and their 3D volumes (15) are constant in time. The same is true for the total number N_s of the magnetic blobs $\mathcal{B}_k^3(t)$ and for their 3D volumes. The frozenness of the magnetic rings and blobs into the plasma flows with zero electrical resistance implies that no collapses or touchings between magnetic rings or blobs can occur during the axisymmetric dynamics of plasma.

5 Safety Factor for the Time-Dependent Axisymmetric Magnetic Fields

In this section, we prove the conservation along the plasma streaklines of the safety factor q for the general helical magnetic field lines on invariant tori $\mathbb{T}^2$ of a non-stationary axisymmetric magnetic field $\mathbf{B}(\mathbf{x}, t)$. The safety factor q was studied previously in [2, 8, 16, 17, 23].

Let for a z-axisymmetric field $\mathbf{B}(\mathbf{x}, t)$ a magnetic field line lie on a torus $\mathbb{T}^2=C\times\mathbb{S}^1$ and is closed and goes (say) m times a long way (along the circle $\mathbb{S}^1$) and n times a short way (along the closed curve C). In [17], the safety factor q for such a closed curve is defined as $q=m/n$. Other definitions of the pitch $p=2\pi q$ for the vortex lines in the fluid equilibria are given in [8, 24].

We propose the following definition of the safety factor of the non-closed helical magnetic field lines on the time-dependent tori $\mathbb{T}^2_{\psi(t)}=C_{\psi(t)}\times\mathbb{S}^1$. Let $\tau(C_{\psi(t)})$ be the period of the corresponding closed trajectory $C_{\psi(t)}$ of the system (13). The generalised safety factor $q(r, z, t)$ of the helices is equal to the increment of the angle φ during one period $\tau(\psi(t))$, divided by 2π:

$$q(r, z, t)=\frac{1}{2\pi}\oint_{C_{\psi(t)}}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\mathrm{d}\tau=\frac{1}{2\pi}\int_0^{\tau(C_{\psi(t)})} r^{-1}(\tau)w[r(\tau), z(\tau), t]\mathrm{d}\tau. \tag{21}$$

Formula (21) defines the safety factor for both the non-closed, infinite helical trajectories of system (13), (14) on the torus $\mathbb{T}^2_{\psi(t)}$ and for the closed ones. The safety factor $q(r, z, t)$ (21) has the same value for all magnetic field lines on a given torus $\mathbb{T}^2_{\psi(t)}$ because the circle integral in (21) does not depend on the starting point. If the number $q(r, z, t)$ is rational $(=m/n)$, then during n periods $\tau(C_{\psi(t)})$ the angle φ increases for $n\cdot 2\pi m/n=2\pi m$. Therefore, the corresponding curve on the torus $\mathbb{T}^2_{\psi(t)}=C_{\psi(t)}\times\mathbb{S}^1$ is closed and makes m complete turns a long way and n turns a short way. Hence our definition (21) being applied to the closed magnetic field lines reduces to the definition given in [17] only for the closed ones. Such closed curves are called "torus knots $K_{m,n}$" [9]. As is known, the rational number m/n is a topological invariant of the torus knot $K_{m,n}$ [9].

5.1 Conservation of the Safety Factor *q*(*r*, *z*, *t*)

Since all magnetic field lines are frozen into the plasma flow, the same is true for the magnetic field knots $K_{m,n}$. Hence, during the plasma dynamics, the main characteristics of the knots $m/n=q(r, z, t)$ is conserved because it is a topological invariant. Therefore, all rational values of the safety factor $q(r, z, t)$ (21) are preserved along the plasma streaklines.

We have assumed that the vector fields $\mathbf{V}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$ are of class C^2. This implies that the function $q(r, z, t)$ (21) is at least continuous. The continuity of $q(r, z, t)$ yields that tori $\mathbb{T}^2_{\psi(t)}$ with rational values of $q(r, z, t)$ are everywhere dense in each closed set $\mathcal{D}_k^3(t)$. Any irrational value of $q(r, z, t)=\xi$ is a limit of certain rational values $q_\ell(r, z, t)=m_\ell/n_\ell$: $\xi=q(r, z, t)=\lim_{\ell\to\infty} m_\ell/n_\ell$. Therefore, the conservation during the plasma dynamics of the rational values $q_\ell(r, z, t) = m_\ell/n_\ell$ and the continuiuty of the function $q(r, z, t)$ (21) yield that all its irrational values $q(r, z, t) = \xi$ also are conserved. Hence the safety factor $q(r, z, t)$ and any functions of it $F(q(r, z, t))$ (e.g. $q^n(r, z, t)$ or $\cos(nq(r, z, t))$ for any integer $n \geq 1$) are conserved along the plasma streaklines during the plasma dynamics.[1] The same is true for the plasma density $\rho(\mathbf{x}, t)$ as a result of the last equation of (3) and for any differentiable function $f(q, \rho)$.

6 Functional Invariants of Plasma Flows

The boundary $\partial D_k(t)$ is a one-dimensional set $\mathcal{C}_k(t)$ that is invariant with respect to the dynamical system (13) and hence satisfies the equation $\psi(r, z, t) = C(t)$. Therefore, for any flux function $\psi(r, z, t)$ ($t = \text{const}$) of class C^3, there exists in the interior of each maximal set $D_k(t)$ at least one point $c_k(t) = (r_k(t), z_k(t))$ of local maximum or minimum of $\psi(r, z, t)$. The point $c_k(t)$ is an equilibrium point of system (13) and is the limit of the neigbouring invariant, small, closed curves $C_{\psi(t)}$, which are transported by the induced diffeomorphisms of the poloidal plane (r, z). Therefore, the points $c_k(t)$ also are trasported with the plasma flow. The same is true for the corresponding magnetic axes $\mathbb{S}_k(t) = c_k(t) \times \mathbb{S}^1$. Let $A_k(t)$ and $B_k(t)$ be the limits of the safety factor $q(r, z, t)$ at the point $c_k(t)$ and at the boundary $\mathcal{C}_k(t) = \partial D_k(t)$, respectively. The limits $A_k(t)$ are finite if the local maximum or minimum $c_k(t)$ of function $\psi(r, z, t)$ is non-degenerate. The limits $B_k(t)$ can be infinite, see [25]. Since the safety factor $q(r, z, t)$ is preserved along the plasma streaklines, we get that its limits $A_k(t)$ and $B_k(t)$ also are preserved. Therefore, the limits do not depend on time t and hence are the new invariants A_k and B_k of the viscous plasma flows. We call them the functional invariants.

Suppose that, for some t, the function $\psi(r, z, t)$ has a non-degenerate local maximum or minimum at a point $c_k(t) = (r_k(t), z_k(t))$. This means that $\partial\psi(c_k(t))/\partial r = 0$, $\partial\psi(c_k(t))\partial z = 0$, and the Hessian

$$\mathcal{H}(c_k(t)) = \frac{\partial^2\psi(c_k(t))}{\partial r^2}\frac{\partial^2\psi(c_k(t))}{\partial z^2} - \left[\frac{\partial^2\psi(c_k(t))}{\partial r\partial z}\right]^2 \tag{22}$$

is positive, $\mathcal{H}(c_k(t)) > 0$. Then the neighbouring curves $\psi(r, z, t) = \psi = \text{const}$ are closed curves $C_{\psi(t)}$ encircling the point $c_k(t)$. The curves $C_{\psi(t)}$ are closed trajectories of system (13). Let $\tau(C_{\psi(t)})$ be their periods. In the limit $\psi(r, z, t) \to \psi(c_k(t))$, the closed curves $C_{\psi(t)}$ ($\psi(r, z, t) = \text{const}$) tend to the equilibrium point $c_k(t)$: $r(\tau) \to r_k(t)$ and $z(\tau) \to z_k(t)$ for all τ. Applying formula (21), we find

$$A_k(t) = \lim_{(r,z)\to(r_k(t),z_k(t))} q(r, z, t) = \tau(\psi(c_k(t)))\frac{w[r_k(t), z_k(t), t]}{2\pi r_k(t)},$$

where $\tau(\psi(c_k(t))) = \lim \tau(C_{\psi(t)})$ when $\psi(t) \to \psi(c_k(t))$.

As is shown above, the value $A_k(t)$ does not depend on t: $A_k(t) = A_k = \text{const}$. Using the formula derived in [25] $\tau(\psi(c_k)) = 2\pi r_k / \sqrt{\mathcal{H}(c_k)}$ for the limit of the periods $\tau(\psi)$ at $\psi \to \psi(c_k(t))$, we get

$$A_k = \frac{w[r_k(t), z_k(t), t]}{\sqrt{\mathcal{H}(c_k(t))}}. \tag{23}$$

Formula (23) defines the explicit form of the new functional invariants A_k of the MHD equations (1)–(3). The Hessians $\mathcal{H}(c_k(t))$ (22) and invariants A_k (23) are independent of the gauge transformations (8).

Remark 1: Formula (23) proves that for any time-dependent axisymmetric magnetic field $\mathbf{B}(\mathbf{x}, t)$, the safety factor $q(\mathbf{x}, t)$ has a finite limit A_k at a magnetic axis $\mathbb{S}_k(t) = c_k(t) \times \mathbb{S}^1$, which means at $\psi(t) \to \psi(c_k(t))$, provided that the non-degeneracy condition $\mathcal{H}(c_k(t)) > 0$ is met. Moffatt mistakenly stated in [16] that for the plasma equilibria the limits A_k are always infinite and the limits B_k in case of blobs $\mathcal{B}_k^3$ are always zero. These two statements have led Moffatt to the erroneous conclusions about the structure of the magnetic field knots.[2]

1 Because of the existence of the gauge transformation (8), the time-dependent flux function $\psi(r, z, t)$ is non-uniquely defined by the magnetic field $\mathbf{B}(r, z, t)$ (9) and is not conserved along the plasma streaklines, see (12). Therefore, the safety factor $q(r, z, t)$ cannot be expressed as a uniquely defined function $q(\psi)$ of the flux function $\psi(r, z, t)$.

2 Namely, on p. 30 of [16]: "It is an intriguing property of this **B**-field that if we take a subset of the **B**-lines consisting of one **B**-line on each toroidal surface, then every torus knot is represented once and only once in this subset, since $p/2\pi$ passes through every rational number m/n once it decreases continuously from infinity to zero". We have demonstrated by counterexamples in [25–27] that this statement is incorrect. In reality, by far not all torus knots are realised as magnetic field knots for the plasma equilibria studied in [16]. For example, for the spheromak Beltrami field, only those torus knots are realised for which the safety factor $q = p/2\pi$ takes rational values from the narrow interval $0.7151 < q < 0.8252$ [25]. The corrected statements are proved in [25].

7 Infinite Families of Integral Invariants

Let $\mathcal{D}_k^3(t_1)$ be a maximal ring or blob, and let $dv = r\,dr\,dz\,d\varphi$ be the 3D volume element. For any $n, m = 0, 1, 2, 3, \ldots$ the formulae

$$J_{kmn} = \int_{\mathcal{D}_k^3(t)} q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t)dv = 2\pi\int_{D_k(t)} q^n(r, z, t)\rho^m(r, z, t)r\,dr\,dz \quad (24)$$

define the infinite family of invariants of the axisymmetric MHD equations (1)–(3). Indeed, the plasma flow preserves all functions $q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t)$); it transforms each set $\mathcal{D}_k^3(t_1)$ into $\mathcal{D}_k^3(t)$ for $t > t_1$ and preserves the 3D volume. Hence the integrals (24) are preserved.

Let us consider also the functionals

$$I_{kn} = \int_{\mathcal{D}_k^3(t)} \cos(nq(\mathbf{x}, t))\rho(\mathbf{x}, t)dv = 2\pi\int_{D_k(t)} \cos(nq(r, z, t))\rho(r, z, t)r\,dr\,dz, \quad (25)$$

which are defined for any integer $n \geq 0$ and form an infinite matrix of invariants of (1)–(3). The matrix I_{kn} (25) has $N = N_r + N_s$ rows because $k = 1, 2, \ldots, N_r + N_s$ and infinitely many colums because $n = 0, 1, 2, 3, \ldots$.

Formulas (24) for $n = m = 0$ and (25) for $n = 0$ imply that the volume and the total mass M_k of each magnetic ring $\mathcal{R}_k^3(t)$ and each magnetic blob $\mathcal{B}_k^3(t)$ are invariants of plasma dynamics.

Each invariant $|I_{kn}|$ (25) is bounded by the total mass M_k of plasma inside the set $\mathcal{D}_k^3(t)$. Indeed, because $|\cos(nq)| \leq 1$, formulas (25) yield $|I_{kn}| \leq \int_{\mathcal{D}_k^3(t)} |\cos(nq(\mathbf{x}, t))|\rho(\mathbf{x}, t)dv \leq \int_{\mathcal{D}_k^3(t)} \rho(\mathbf{x}, t)dv = M_k$.

The mutual functional independence of invariants I_{kn} follows from that for the functions $\cos nq$ for $n = 1, 2, 3, \ldots$.

The conservation of functions $q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t)$) along the plasma streaklines is equivalent to the equation

$$\partial[q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t))]/\partial t + \sum_{i=1}^{3} V_i(\mathbf{x}, t)\partial[q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t))]/\partial x_i = 0,$$

where $V_i(\mathbf{x}, t)$ are components of the plasma velocity. This equation, together with equation $\operatorname{div}\mathbf{V} = 0$, defines an infinite family of conservation laws

$$\frac{\partial[q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t))]}{\partial t} + \sum_{i=1}^{3} \frac{\partial[V_i(\mathbf{x}, t)q^n(\mathbf{x}, t)\rho^m(\mathbf{x}, t))]}{\partial x_i} = 0,$$

where $n, m = 0, 1, 2, 3, \ldots$.

7.1 Functional Independence of the New Integral Invariants from the Magnetic Helicity

Using formula (21) for the safety factor $q(r, z, t)$, we present the conserved quantities J_{k0n} (24) in the form

$$J_{k0n} = \frac{1}{(2\pi)^{n-1}}\int_{D_k(t)} \left[\oint_{C_{\psi(t)}} r^{-1}w\,d\tau\right]^n r\,dr\,dz, \quad (26)$$

where $C_{\psi(t)}$ is the closed trajectory of system (12) in the time τ with the initial data (r, z) for $t = \text{const}$. Magnetic helicity $H_k(t)$ (18) for the same magnetic ring or blob $\mathcal{D}_k^3(t)$ is

$$H_k(t) = 2\pi\int_{D_k(t)} [2\psi(r, z, t) - b_k(t)]w(r, z, t)dr\,dz, \quad (27)$$

where $b_k(t)$ is defined by the equation $\psi(r, z, t) = b_k(t)$, the boundary curve $\mathcal{C}_k(t)$ of the domain $D_k(t)$.

Let us show that the new invariants (26) are functionally independent of the helicity (27). Indeed, for any magnetic ring or blob $\mathcal{D}_k^3(t)$, the integral invariants (26) as functions of the magnetic field $\mathbf{B}$ are mutually functionally independent because the power functions q^n and q^m for $n \neq m$ are independent. Therefore, if for a fixed k all integrals (26) with arbitrary $n \geq 1$ were functionally dependent on the helicity $H_k(t)$ (27), then J_{k0n} would have been mutually functionally dependent on each other, but they are not. This proves that the integral invariants J_{k0n} for all $n \geq 1$ with possibly one exception are functionally independent of the magnetic helicity H_k (27).

The integrals J_{kmn} (24) for $m \geq 1$ and I_{kn} (25) depend on plasma density ρ and on the safety factor q. These integrals for the non-constant plasma density $\rho(r, z, t)$ are functionally independent of the helicity $H_k(t)$ by the same reasons as above and also because formula (27) does not contain $\rho(r, z, t)$.

8 Conclusion

The following main results were obtained in this article:

1. For the axisymmetric dynamics of viscous plasmas with zero electrical resistance, we have proved the existence of magnetic rings $\mathcal{R}_k(t)$ and magnetic blobs $\mathcal{B}_m(t)$ which are frozen into the plasma flows and therefore cannot intersect each other during the plasma dynamics.
2. We constructed the family of functional invariants A_k which are presented by the explicit formula (23).

3. We presented infinitely many integral invariants J_{knm} (24) and I_{kn} (25) and proved that they are functionally independent of the magnetic helicity (4), (27).
4. The axisymmetric dynamics of plasmas with zero electrical resistance between two given states is possible only if the corresponding total numbers of magnetic rings N_r are equal (the same for the total numbers of magnetic blobs N_s) and all new invariants A_k, J_{kmn}, I_{kn} for them coincide.

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References

[1] W. A. Newcomb, Ann. Phys. **3**, 347 (1958).
[2] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, Cambridge 1997.
[3] O. I. Bogoyavlenskij, Phys. Lett. A **307**, 281 (2003).
[4] O. I. Bogoyavlenskij, Phys. Rev. E **66**, 5640 (2002).
[5] O. I. Bogoyavlenskij, Phys. Rev. Lett. **84**, 1914 (2000).
[6] O. I. Bogoyavlenskij, Phys. Rev. E **62**, 8616 (2000).
[7] S. Grozdanov, D. M. Hofman, and N. Iqbal, Phys. Rev. D **95**, 096003 (2017).
[8] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[9] R. H. Crowell and R. H. Fox, Introduction to Knot Theory, Ginn and Company, Boston 1963.
[10] L. Woltjer, Proc. Nat. Acad. Sci. **44**, 489 (1958).
[11] M. A. Berger, Plasma Phys. Control. Fusion **41**, B167 (1999).
[12] D. Serre, Physica **13D**, 105 (1984).
[13] E. A. Kudryavtseva, Math. Notes **95**, 877 (2014).
[14] E. A. Kudryavtseva, Math. Notes **99**, 611 (2016).
[15] A. Enciso, D. Peralta-Salas, and F. Torres de Lizaur, PNAS **113** 2035 (2016).
[16] H. K. Moffatt, Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Press, Cambridge 1978.
[17] C. Mercier, Fusion Nucleaire **4**, 213 (1964).
[18] D. W. Kerst, J. Nucl. Energy, Part C **4**, 253 (1962).
[19] K. J. Whiteman, Rep. Prog. Phys. **40**, 1033 (1977).
[20] A. H. Boozer, Phys. Fluids **26**, 1286 (1983).
[21] A. H. Boozer, Rev. Mod. Phys. **76**, 1071 (2005).
[22] E. L. Ince, Ordinary Differential Equations, Dover Publications, New York 1956.
[23] J. R. Cary and R. G. Littlejohn, Ann. Phys. **151**, 1 (1983).
[24] W. M. Hicks, Phil. Trans. Roy. Soc. London A **192**, 33 (1899).
[25] O. Bogoyavlenskij, Phys. Rev. E, **95**, 043104, 14 (2017).
[26] O. Bogoyavlenskij, J. Math. Anal. Appl. **450**, 21 (2017).
[27] O. Bogoyavlenskij, J. Math. Phys. **5**, 013101-01 (2017).