CAP EXAM - USEFUL INFORMATION

January 2005

MECHANICS

Elementary Approach:

1. **Describe the kinematics (a) Co-ordinates**
   (b) Constraints

2. **Diagram with all of the forces**

3. **Statics:** $\sum F_x = 0; \quad \sum F_y = 0; \quad \sum \underline{\tau} = 0$

4. **Rigid Body-Free**

   (a) Motion of C of M: $\underline{F} = m\underline{\tilde{a}}$

   (b) Rotation about C of M: $\tilde{\tau} = \underline{\tilde{I}}\ \underline{\tilde{\alpha}}$

5. **Rigid Body-Fixed Point 0**

   Rotation about 0 $\underline{\tau}_0 = \underline{I}_0 \underline{\alpha}_0$

6. **Use energy if "complicated"**

7. **Rotating System - use Euler's Equation**

$$\underline{\tau} = \frac{d\underline{\underline{L}}}{dt} \text{ in the rest frame}$$

becomes $\underline{\tau} = \frac{d\underline{\underline{L}}}{dt} + \underline{\omega} \times \underline{L}$ **in the rotating frame.**

In the rotating frame, use principal axes as co-ordinates.

Lagrangian Mechanics:

1. Choose co-ordinates q_1, q_2 etc.

2. Express kinetic energy (T) in terms of the co-ordinates and their time derivatives $\dot{q}_1$, $\dot{q}_2$ etc.

3. Express the potential energy (V) in terms of the co-ordinates.

4. Write down the Lagrangian L = T - V.

5. Use the Lagrange equation

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = 0$$

for each of the co-ordinates q_i as the equations of motion.

ELECTRICITY AND MAGNETISM

1. Gauss' Law: $\oint \underline{E} \cdot d\,\underline{A} = Q/\varepsilon_0$ where Q is the charge enclosed within the closed surface. In a dielectric medium $\oint \underline{D} \cdot d\,\underline{A} = Q/\varepsilon_0$ $(D = \varepsilon E)$

2. Potentials:

Point charge: $V = Q/4\pi\varepsilon_0 r$

Charged sphere, radius a

Surface potential: $V = Q/4\pi\varepsilon_0 a$

r > a: $V = Q/4\pi\varepsilon_0 r$

Electric field: $\begin{cases} \underline{E} = -\nabla V \\ E_x = -dV/dx \text{ (one dimension)} \end{cases}$

3. Magnetic Field:

Amperès Law $\oint B \cdot dl = \mu_0 I$

Long straight wire $B = \dfrac{\mu_0 I}{2\pi r}$ (tangential)

Lorentz Force $d\,\underline{F} = dq\,\underline{v} \times \underline{B}$

or $d\,\underline{F} = I\,d\,\underline{l} \times \underline{B}$

Lenz's Law $V = -\dfrac{d\phi}{dt}$

Maxwell's Eqns. $\text{div}\,\underline{D} = \rho$; $\text{div}\,\underline{B} = 0$

$\text{curl}\,\underline{B} = \mu_0 \underline{J}$; $\text{curl}\,\underline{E} = -\dfrac{d\,\underline{B}}{dt}$

THERMODYNAMICS

Ideal Gas:

$PV = nRT$ – always

PV^{γ} = const. for an adiabatic change

$U = fn(T)$ only

$dU = C_V dT; \ dH = C_P dT$

Central Equation:

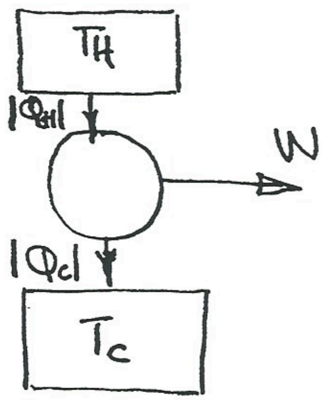
with 1st Law: $dU = dQ - dW$
with 2nd law: $dU = TdS - PdV \ (+ HdM + Ed\wp + FdL + EdZ)$

Maxwell's Relations:

$$(\partial T/\partial V)_S = -(\partial P/\partial S)_V; \ (\partial T/\partial P)_S = (\partial V/\partial S)_P$$

$$(\partial S/\partial V)_T = (\partial P/\partial T)_V; \ (\partial S/\partial P)_T = -(\partial V/\partial T)_P$$

Carnot Engine:



$$W = |Q_H| - |Q_C|; \eta = \frac{W}{|Q_H|}$$

Carnot theorem:- reversible heat engines

$$\frac{|Q_H|}{|Q_C|} = \frac{T_H}{T_C}$$

$$\therefore \ \frac{Q_H}{T_H} + \frac{Q_C}{T_C} = 0$$

General reversible cycle:

$$\oint \frac{dQ}{T} = 0 \text{ or } \oint dS = 0$$

Note - For the reversible (Carnot) heat engine, the entropy of the total system is conserved.

SPECIAL THEORY OF RELATIVITY

Lorentz-Einstein Transformations

S is a frame at rest. S' has velocity v in the x direction

$x' = \gamma(x - vt)$ $\quad\quad$ $x = \gamma(x' + vt')$

$t' = \gamma(t - xv/c^2)$ $\quad\quad$ $t = \gamma(t' + x'v/c^2)$

$y' = y$ $\quad\quad$ $\gamma = 1\sqrt{1 - v^2/c^2}$

$z' = z$

Lorentz contraction

L is the length of a ruler at rest in S

$$L = x_2 - x_1$$

L' is the length measured in S'; it is $x_2' - x_1'$ with x_1' and x_2' measured at the same time t'. Use,

$$x_2 = \gamma(x_2' + vt')$$

$$x_1 = \gamma(x_1' + vt')$$

$$\therefore L' = L/\gamma$$

the moving ruler is measured to be shorter.

Time Dilation

A clock in S is at rest. It measures a time interval $T = t_2 - t_1$. Its position x is constant. The equivalent interval in S' is

$$T' = t_2' - t_1' = \gamma T \text{ from above.}$$

That is $T < T'$. The time interval registered by the moving clock is less than the time interval in the frame S'. The moving clock runs slowly.

Velocities

$$u_x' = x'/t' = (u_x - v)/(1 - u_x v/c^2)$$

$$u_y' = y'/t' = u_y/\gamma(1 - u_x v/c^2)$$

Doppler Shift

$$\omega' = \omega\sqrt{(1 - v/c)/(1 + v/c)} \quad \text{receding}$$

$$\omega' = \omega\sqrt{(1 + v/c)/(1 - v/c)} \quad \text{approaching}$$

Dynamics

Particles: $E^2 = E_0^2 + p^2c^2$

with $E = mc^2 = \gamma m_0 c^2$; $E_0 = m_0 c^2$; $p = mv$

Photon, Neutrino $m_0 = 0$; $E_0 = 0$; $E = pc$

Conserve momentum and energy.

STATISTICAL MECHANICS

Microstates: Defined by giving the state of each particle in the system. Every accessible microstate is equally probable.

For a system composed of two parts, with numbers of accessible microstates Ω_1 and Ω_2, for the system $\Omega = \Omega_1\Omega_2$. The temperature of a system is $\frac{1}{k_B T} = \frac{d\ln\Omega(E)}{dE}$.

Boltzmann Factor: The probability that a system is in a microstate r when in thermal equilibrium with a reservoir at temperature T is,

$$P_r = e^{-\beta E_r} / \sum_r e^{-\beta E_r}$$

Alternatively,

$$dP(E) = g(E)dEe^{-\beta E} / \int g(E)dEe^{-\beta E}$$

where $\beta = 1/k_B T$, E_r is the energy of state r and g(E) is the density of states.

Mean Value: For a system in equilibrium at temperature T

$$\overline{a} = \sum_r a_r P_r.$$

Partition Function: $Z = \sum_r e^{-\beta E_r}$

Then,

$$\overline{E} = -\, d\ln Z / d\beta$$

$$P = kT(d\ln Z / dV)$$

$$F = -kT\ln Z$$

$$C_V = -(d\beta / dT)(d^2 \ln Z / d\beta^2)$$

Chemical Potential $\mu = -kT(d\ln Z/dN)$

Classical and Quantum Gases:

Classical: All microstates are allowed. Bose-Einstein only one of (0110), (1010), (0011).

Fermi-Dirac: (0012) is forbidden
(1347) is good.

Mean occupation: Classical: $f_s \propto e^{-\beta\varepsilon_s}$

Bose: $$f_s = \frac{1}{e^{\beta(\varepsilon_s - \mu)} - 1}$$

Fermi: $$f_s = \frac{1}{e^{\beta(\varepsilon_s - \mu)} + 1}$$

QUANTUM MECHANICS

Schrödinger's Equation:

$$\hat{H}u = Eu \quad \text{(time-independent potential)}$$

$$u(r,t) = u(r)e^{-iEt/\hbar}$$

$$\psi = \sum_n c_n u_n; \quad \sum c_n^2 = 1$$

$$c_n = \int u_n^* \psi d\tau$$

New Basis

$$v_n = \sum c_m u_m$$

$$\int v_m^* v_n d\tau = \delta_{m,n}$$

Operators

$$\hat{x} = x; \quad p_x = -i\hbar \frac{d}{dx}, \quad \hat{H} = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V$$

Boundary Conditions

$$V = \infty \qquad u_1 = u_2$$

$$V = \text{step} \quad u_1 = u_2 \text{ and } \frac{du_1}{dx} = \frac{du_2}{dx}$$

Time Dependence

e.g. $\psi(x,t) = \frac{1}{\sqrt{2}}\left(u_\alpha e^{-iE_\alpha t/\hbar} + u_\beta e^{-iE_\beta t/\hbar}\right)$

Hence interference.

Perturbation Theory

$$E_m = E_m^0 + H'_{mm} + \sum_k \frac{\left|H'_{mk}\right|^2}{E_m^0 - E_k^0}$$

where $H'_{mk} = \int u_m^* H' u_k \, d\tau$

$$u_m = u_m^0 + \sum \frac{H'_{km} u_k^0}{E_m^0 - E_k^0}$$