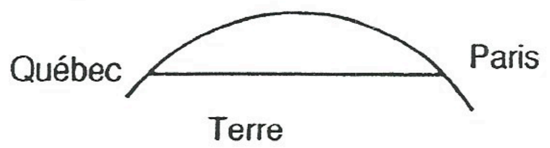
CAP University Prize Examination
Thursday, February 8, 2001, 14 :00-17 :00

Introduction

1. Calculators are allowed
2. Answer each question in a different booklet, with the question number, your name and that of your university on the first page.
3. The value attributed to each question takes into account the length and difficulty of the problem.
4. You are not expected to answer all questions. Please try as many as you can.

I(10)

We design a long range transport system by digging straigh line tunnels between surface destinations. Gravitational attraction provides all the needed acceleration and deceleration. High vacuum and magnetic levitation guarantee that no friction acts on the trains in the tunnels.



How long would it take to travel from Québec to Paris, from Québec to Toronto?

II(10)

Imagine a thermodynamic cycle followed in the 123 direction, where 12 is an adiabatic process, 23 an isothermic one and 31 an isochore decompression. This cycle represents the transformations of one kmole of a real gas in a machine (it remains gaseous).

1. Draw this cycle in the p-V plane
2. If the van der Walls equation is a good representation of this gas, what is the efficiency of this machine
3. Which of the two real gas effects affects most this efficiency, the finite volume of the molecules or the interaction between the molecules?

III(10)

It is possible to view a dielectric medium as made up of atoms on a lattice. Each atom has one electron of charge e bound to the site by a potential which we approximate by an harmonic potential of frequency ω in a Hamiltonian operator H_0. There are N atoms per unit volume. The dielectric medium is placed in an electric field $E\hat{z}$ (in the Oz direction). The field induces a polarisation (dipole) $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ in the medium, where χ_e is the electric susceptibility of the medium.

1. Give an expression for χ_e as a function of the parameters of the problem.
2. Does the susceptibility χ_e increase or decrease when the frequency ω of the harmonic potential increase and why is it so .

IV(10)

A system contains N identical particles in a one dimensional space. They undergo interactions, which global result can be represented by an harmonic potential $V = \dfrac{m\omega^2 x^2}{2}$.

If the particles are in a quantum regime, they can occupy the states of energy $\varepsilon_n = (n + 1/2)\hbar\omega,\ n = 0,1,2,3...$

If the particles are in a classical regime, their energy is given by

$$\varepsilon = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

1. Find the (classical) partition function for each regime.
2. Write an expression for the specific heat in each regime?
3. Do these specific heats differ at low temperature, at high temperature?

V(10)

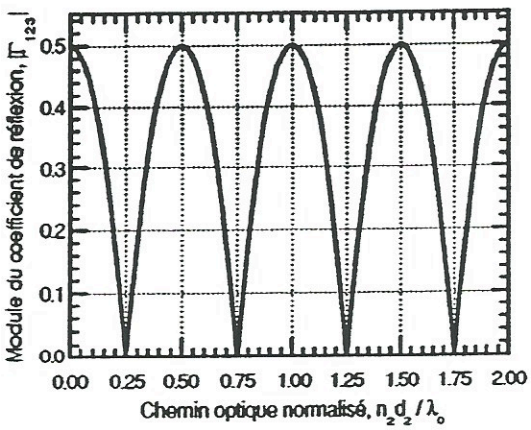
An observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars. That is, in any solid angle $d\Omega$ he sees $dN = N(d\Omega/4\pi)$ stars, where N is the total number of stars he can see.

Suppose that another observer is moving at a relativistic velocity β in the $\hat{x}$ direction (rest frame system S'). What is the distribution of stars seen by this observer? Specifically, what is the distribution function $P(\theta',\phi')$ such that the number of stars seen by this observer in his solid angle $d\Omega'$ is $P(\theta',\phi')\,d\Omega'$? Check to see that $\int_{sphere} P(\theta',\phi')d\Omega' = N$ and that $P(\theta',\phi') \to \dfrac{N}{4\pi}$ as $\beta \to 0$.

Where will the observer see the stars *bunch up*?

VI(10)

You are asked to interpret the results obtained as a thin layer of material of index n_2 (n_2>1) is deposited on a very thick substrate (semi infinite in fact) of index n_3 (n_3>1). The surrounding medium is a vacuum (n_1=1). While depositing our layer (medium 2), the modulus of the reflection coefficient, $|\Gamma|$, (ratio of the amplitude of reflected wave over the amplitude of the incoming wave) was monitored as a function of the thickness of the layer. The result is reported on the figure where the vertical axis should read "Modulus of reflection coefficient $|\Gamma|$" and the horizontal should read "Normalised optical path $n_2 d_2 / \lambda_2$".



a. Using the graph, determine first the index n_3 of the substrate and then the index n_2 of the material in the thin layer.

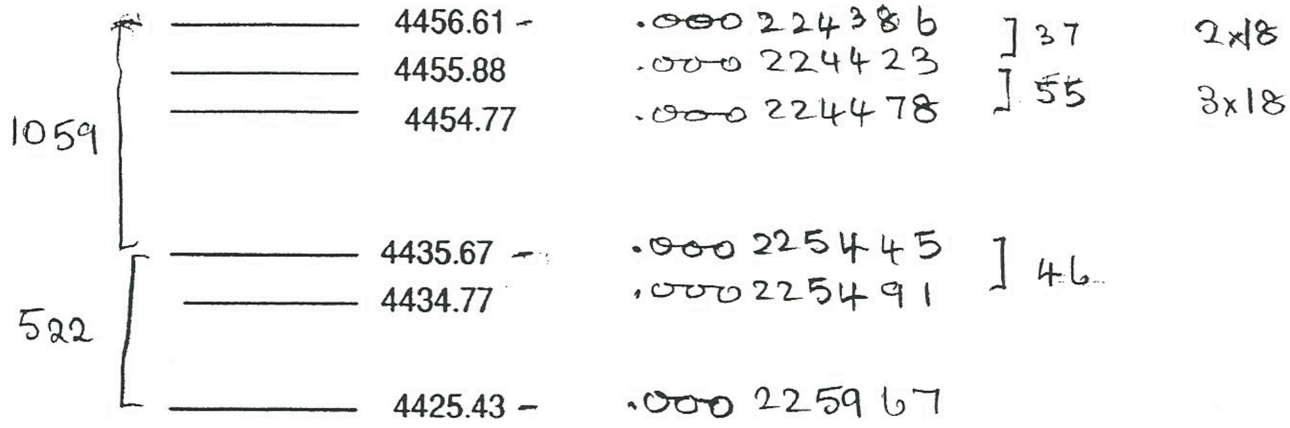
b. If the thickness of medium 2 equals $d_2 = \lambda_2 / 8$, what is the ratio of the (outside) stationary wave. Use the graphics to avoid a long calculation.

c. What would be the sign of Γ for $d_2 = \lambda_2 / 2$? Why?

VII(8)

The following figure shows the observed lines, with λ in angströms, in the spectrum of a certain atom of intermediary Z. These lines correspond to all the possible (optically allowed) transitions between the levels of two multiplets.

Determine the quantum numbers *LSJ* characterizing these multiplets and their levels. Explain your results.



VIII(7)

This problem finds some applications in biophysics where it can be used to model some neurophysical processes. A spherical current source is embedded in a medium where the conductivity behaves as $1/r^2$ with respect to the center of the source. Determine the electrical potential and the charge density everywhere outside the source.

IX(10)

1. Write down the law of radioactive decay. Define the half-life and mean life of a radioactive nucleus and obtain the relation between them.
2. The nucleus ^{87}Rb (Z=37) decays into the ground state of ^{87}Sr (Z=38), with a half-life of 4.7×10^{10} years and a maximum energy for the β of 272 keV. Discuss briefly the difficulties you might encounter in attempting to measure this half-life.
3. Five samples of chondritic meteorites are found to have the following proportions of ^{87}Rb, ^{87}Sr et ^{86}Sr.

Meteorites	^{87}Rb/^{86}Sr	^{87}Sr/^{86}Sr
Modoc	0.86	0.757
Homestead	0.8	0.751
Bruderheim	0.72	0.747
Kyushu	0.6	0.739
Bath Furnace	0.09	0.706

Given that the nucleus ^{86}Sr is not a daughter product of any long-lived radioactive nucleus, show that these data are consistent with a common primordial ratio $^{87}Sr/^{86}Sr$ and a common age for all these meteorites and find that age.

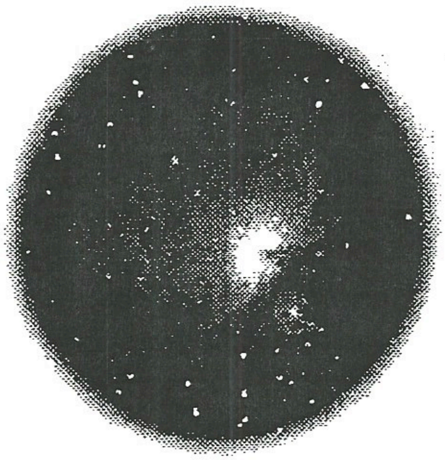
X(10)

The Orion nebula is 1600 light years away from the Sun and is completely ionised. Its average temperature is 8500 K and its electronic density is 2000 cm^{-3}. Its diameter is estimated to be 1.6 light year (one light year is 365 light days).

a) Suppose that nebula is made of non collisional plasma (a good approximation), what is the cutoff frequency f_p (in Hz) for electromagnetic waves propagating in this nebula.

b) Two light pulses, P_1 and P_2, with carrier frequencies f_1=15 GHz and f_2 =20 GHz go through the nebula. If P_1 and P_2 come into the nebula at the same time, what will be the delay (in seconds) between the two signals when they come out of the nebula. (Be more intelligent than your calculator; the binomial approximation could be useful...)

c) Refer to b). At the carrier frequency f_1 , what is $\Delta\varepsilon_r (= \varepsilon_{rp} - 1)$, the difference between the relative permitivity of the plasma and that of the vacuum.



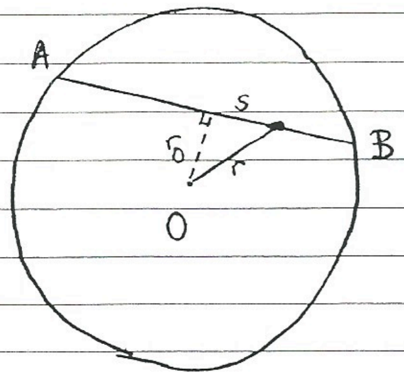
From Ian Turner.

CAP Exam 2001 — Question I

The potential energy for a particle of mass, m, a distance r from the centre of the earth is

$$V = \frac{1}{2} m \frac{g}{R} r^2$$

where g = gravitational field at surface of earth
R = radius of earth.



Consider the chord AB through the earth.
Let r_0 be the closest distance of the chord to the centre of the earth.
Let s be the distance along the chord, measured from the midpoint of the chord.
Then

$$r^2 = s^2 + r_0^2$$

The kinetic energy of a particle of mass m travelling along AB is

$$T = \frac{1}{2} m \dot{s}^2$$

Lagrangian:

$$L = T - V$$

$$= \frac{1}{2} m \dot{s}^2 - \frac{1}{2} m \frac{g}{R} r^2$$

$$= \frac{1}{2} m \dot{s}^2 - \frac{1}{2} m \frac{g}{R} (s^2 + r_0^2)$$

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

$$m \ddot{s} + m \frac{g}{R} s = 0$$

$$\ddot{s} = -\frac{g}{R} s$$

The equation of simple harmonic motion.

The particle undergoes simple harmonic motion with frequency

$$\omega = \sqrt{g/R}$$

The time τ_0 for the particle to slide from one end of the tunnel to the other is half a period

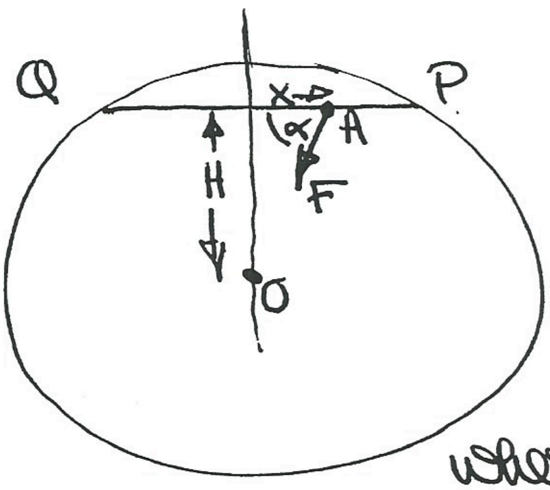
$$\tau_0 = \frac{\pi}{\omega} = \pi\sqrt{R/g} = 42.2 \text{ min}$$

Notice this result is independent of the precise location of the chord. Each and every chord yields the same period of oscillation

Thus the time to go from Quebec to Toronto is the same as from Quebec to Paris, namely 42.2 min.

CAP Exam

2001, #1



Consider the chord defined by the dimension H.

For the displacement x shown, with OA = r

$$F = -\frac{Gm}{r^2}\left(\frac{r}{R}\right)^3 M \ \hat{\underline{r}}$$

where R is the earth radius
M is the earth mass
m is the train mass.

But, for r = R, $F = -mg\,\hat{\underline{r}}$

$$\therefore \frac{GM}{R^2} = g.$$

$$\therefore F = -mg\left(\frac{r}{R}\right)\hat{\underline{r}}$$

x-component is $F = -mg\left(\frac{r}{R}\right)\cos\alpha$

$$= -mg\left(\frac{\cancel{r}}{R}\right)\left(\frac{x}{\cancel{r}}\right)$$

Equation of motion: $\boxed{m\ddot{x} = -m\left(\frac{g}{R}\right)x}$

S.H.O. with $T = 2\pi\sqrt{\frac{R}{g}}$

Travel time $= \frac{T}{2} = \pi\sqrt{R/g} = \pi\sqrt{\frac{6.38\times10^6}{9.8}}$

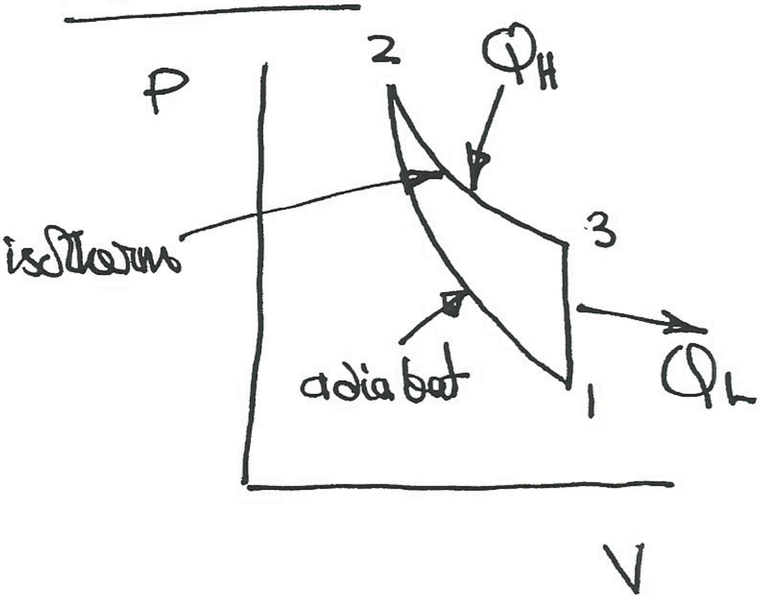
$\boxed{\text{Time} = 42.2 \text{ mins}}$ For all chords.

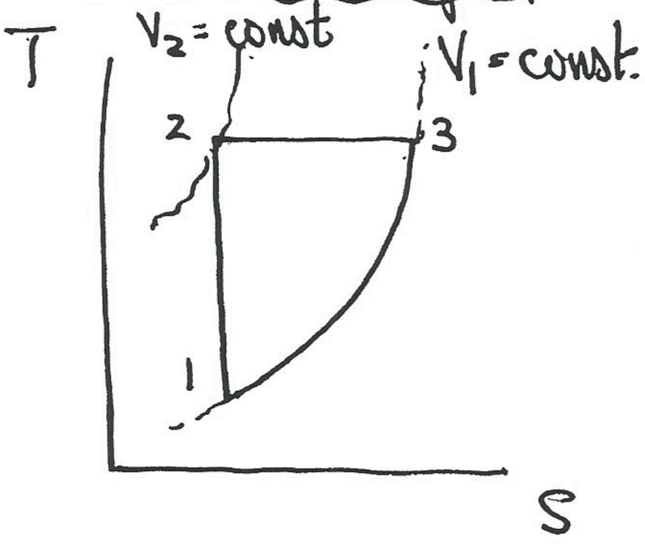
(TPH)

CAP EXAM.

2001 #2

From Don Taylor.





Choose V_1 and T_1 ; then T_2.

Find V_2 (1 and 2 are on an adiabat.)

$$T\,dS = C_V\,dT + T\left(\frac{\partial P}{\partial T}\right)_V dV$$

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \quad \text{for the vdW gas}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{R}{V-b}.$$

$$\therefore T\,dS = C_V\,dT + \frac{RT}{V-b}\,dV$$

$$\text{and} \quad dS = \frac{C_V}{T}\,dT + \frac{R}{V-b}\,dV$$

$\therefore$ Along the adiabat:

$$\int_1^2 \frac{C_V}{T} dT + \int_1^2 \frac{R}{V-b} dV = 0.$$

$$\therefore \int_1^2 \frac{C_V}{T} dT + R \ln\left(\frac{V_2-b}{V_1-b}\right) = 0.$$

Hence V_2

The Heat Engine.

$$Q_H = \int_2^3 T ds = \int_2^3 C_V dT + RT_2 \int_2^3 \frac{dV}{V-b}$$

$$= 0 + RT_2 \left[\ln\left(\frac{V_1-b}{V_2-b}\right) \right]$$

$$= T_2 \int_1^2 \frac{C_V}{T} dT$$

$$Q_L = -\int_3^1 T ds = -\int_3^1 C_V dT + \underset{\text{isochore.}}{0}$$

$$Q_L = \int_1^2 C_V dT.$$

$$\therefore \eta = 1 - \frac{Q_L}{Q_H} = 1 - \frac{\int_1^2 C_V dT}{T_2 \int_1^2 \frac{C_V}{T} dT}$$

For the van der Waals, C_v is a function only of temperature. However it can be a function of a and b.

from Eugene Zaremba.

CAP Exam 2001 #3

III 1. It's unclear what is expected. Anyway...

$\pm e$ charges separate a distance x in an electric field E: $eE = kx$

↳ harmonic force constant $k = m\omega^2$

$$\therefore \; x = \frac{eE}{m\omega^2}$$

Dipole moment: $p = ex = \dfrac{e^2}{m\omega^2}E = \alpha E$

$$\alpha = \frac{e^2}{m\omega^2} = \text{atomic polarizability.}$$

In terms of the atomic dipole, the polarization is

$$P = np = n\alpha E$$

Now, E is the local field. This is where the ambiguity arises. Assuming the local field is E, then

$$\epsilon_0 \chi_e E = n\alpha E$$

$$\chi_e = \frac{n\alpha}{\epsilon_0} = \frac{ne^2}{m\omega^2\epsilon_0}$$

The local field E_{loc} is not E, the macroscopic E field. The usual argument gives

$$E_{loc} = E + \frac{P}{3\epsilon_0} = \left(1 + \tfrac{1}{3}\chi_e\right)E$$

$$P = n\alpha E_{loc} = n\alpha\left(1+\tfrac{1}{3}\chi_e\right)E$$

$$\therefore \; \epsilon_0 \chi_e \cancel{E} = n\alpha\left(1 + \tfrac{1}{3}\chi_e\right)\cancel{E}$$

$$\chi_e = \frac{n\alpha}{\epsilon_0} + \frac{n\alpha}{3\epsilon_0}\chi_e$$

$$\chi_e = \frac{n\alpha/\epsilon_0}{1 - n\alpha/3\epsilon_0}$$

~~For~~

For $n\alpha/\epsilon_0 \ll 1$, get back $\chi_e \simeq \frac{n\alpha}{\epsilon_0} = \frac{ne^2}{m\omega^2\epsilon_0}$

From this expression, $\chi_e \downarrow$ as $\omega \uparrow$. This is simply due to the fact that the polarizability decreases as the electron becomes more tightly bound.

Defining $\omega_p^2 = \frac{ne^2}{m\epsilon_0}$, the full expression gives

$$\chi_e = \frac{\omega_p^2/\omega^2}{1 - \frac{1}{3}\frac{\omega_p^2}{\omega^2}}$$

$$= \frac{\omega_p^2}{\omega^2 - \frac{1}{3}\omega_p^2}$$

For $\omega > \frac{\omega_p}{\sqrt{3}}$, χ_e is positive and diverges at ~~ω_p~~ $\omega = \omega_p/\sqrt{2}$. Thus, as ω is reduced from above ω_p, χ_e increases. The divergence is a signature of a polarization catastrophe (phase transition).

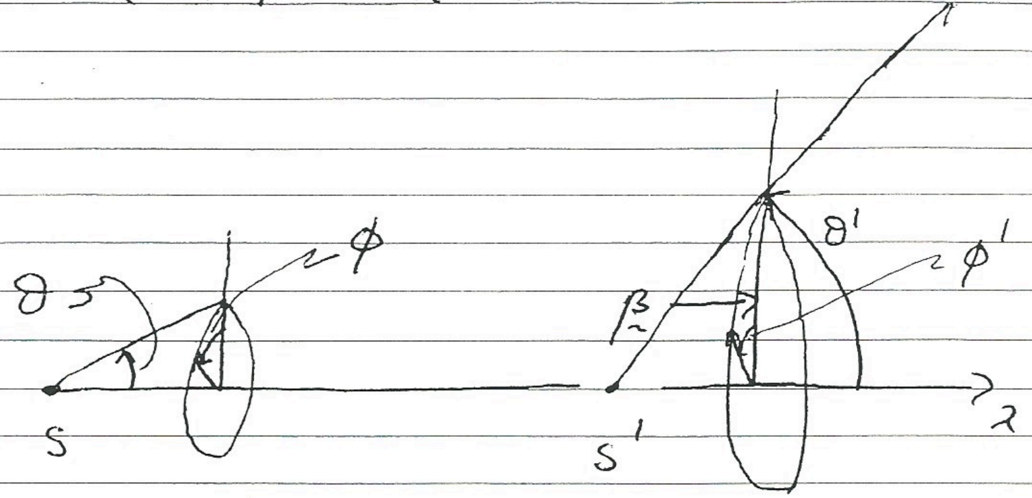
I find it hard to believe this is what students were supposed to do!

[illegible]

From Richard Henrikson.

C. A. P. 2001

#5



Now:

for S

$$dN = \frac{d\Omega}{4\pi} N \qquad \text{in solid angle } d\Omega$$

Now for S'

$$\underline{d\Omega' = d\Omega\, \gamma^2 (1+\beta\cos\theta')^2}$$

<u>Since</u>:

$$d\Omega' \equiv \sin\theta'\, d\theta'\, d\phi'$$

and $d\phi' = d\phi$ in the perpendicular plane

and $\cos\theta' = \dfrac{\cos\theta - \beta}{1-\beta\cos\theta}$ or: $\cos\theta = \dfrac{\cos\theta' + \beta}{1+\beta\cos\theta'}$

by Aberration formulae

$$\Rightarrow \boxed{\sin\theta'\, d\theta' = \sin\theta\, d\theta\, \gamma^2 (1+\beta\cos\theta')^2}$$

hence:

Since N is the same for S'

$$\therefore \quad dN = \frac{N}{4\pi} \frac{d\Omega'}{\gamma^2 (1+\beta\cos\theta')^2}$$

i.e. $$\boxed{\frac{dN}{d\Omega'} = \frac{dN}{d\Omega} \frac{1}{\gamma^2(1+\beta\cos\theta')^2}}$$

(a) $\therefore$ Large as $\theta' \to \pi$ $\quad \beta \to 1$

(b) $$\int \frac{dN}{d\Omega'} d\Omega' = \frac{dN}{d\Omega} \int \frac{1}{\gamma^2(1+\beta\cos\theta')^2} \sin\theta' d\theta' d\phi$$

$$= \frac{N}{4\pi\gamma^2} \cdot 2\pi \cdot \int_{-1}^{1} \frac{dx}{(1+\beta x)^2}$$

$$= \frac{N}{2\gamma^2} \cdot \frac{1}{\beta} \cdot \left\{ \frac{1}{1-\beta} - \frac{1}{1+\beta} \right\}$$

$$= \frac{N}{2\gamma^2\beta} \cdot 2\beta\gamma^2 = \underline{\underline{N}}$$

From Malcolm Stott.

[redacted] 2001 #7

	λ Å	$\varepsilon = h\nu$ eV	$\Delta\varepsilon$ eV
1	4456.61	2.78687	
			.00046
2	4455.88	2.78733	
			.00069
3	4454.77	2.78802	
			.01316
4	4435.67	2.80003	
			.00057
5	4434.77	2.80060	
			.00648
6	4425.43	2.80651	

Intermediate Z suggests LS coupling to account for the spin orbit interaction.

The spin-orbit energy which splits a multiplet is then $\langle A\, \hat{L}.\hat{S} \rangle = \langle A(\hat{J}^2 - \hat{L}^2 - \hat{S}^2) \rangle$

$$= A\hbar^2 (J(J+1) - L(L+1) - S(S+1))$$

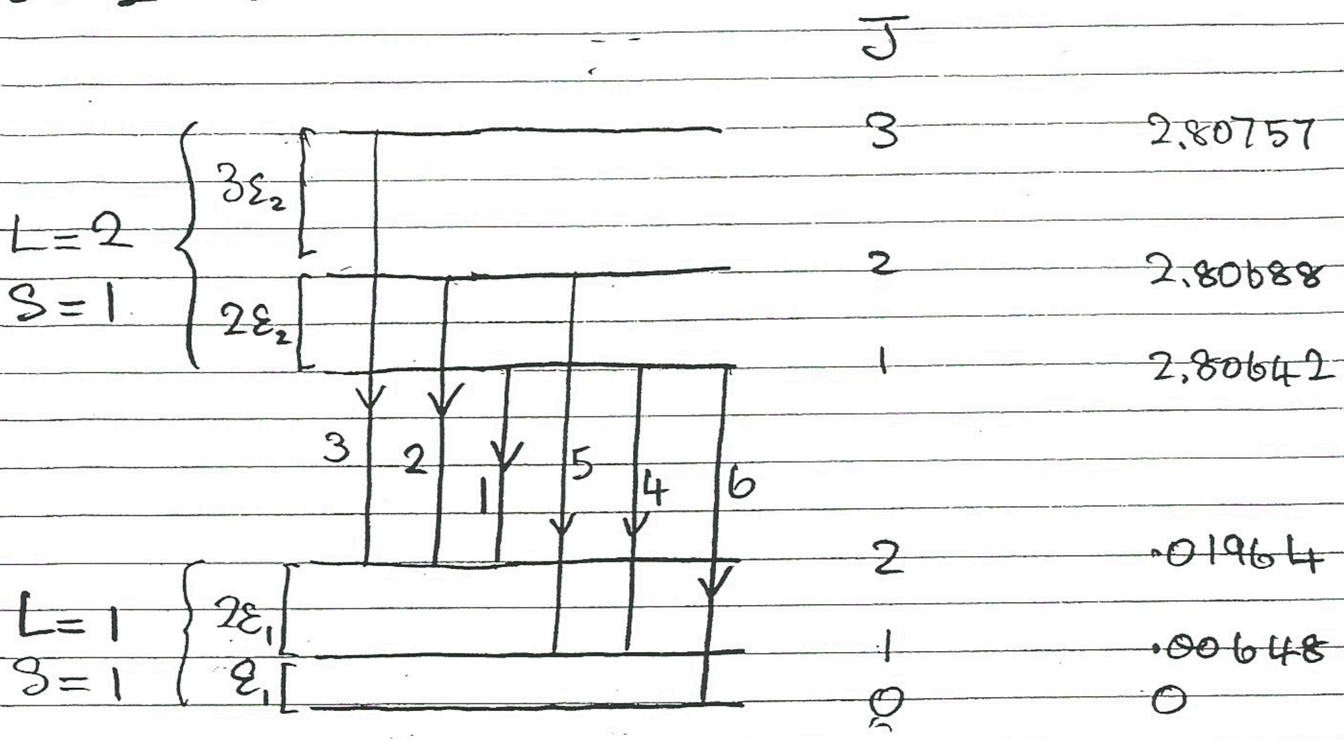
For a given L and S $\quad J = L+S, L+S-1 \cdots |L-S|$

The selection rules for optical (dipole) transitions between states of different multiplets are $\Delta S = 0$, $\Delta J = \pm 1, 0$ (and $\Delta L = \pm 1, 0$)

The first step with the data is to calculate the energies of the spectrum lines and to look for a pattern in the energy differences. You note

$$\frac{\Delta\varepsilon_{12}}{\Delta\varepsilon_{23}} \simeq \frac{2}{3}, \quad \frac{\Delta\varepsilon_{14}}{\Delta\varepsilon_{46}} \simeq \frac{2}{1}$$

$S = 0$ or $\frac{1}{2}$ will not give enough lines. For $S = 1$ we have



This choice $L=2$, $L=1$ fits the given pattern in except for the energy difference $\Delta\mathcal{E}_{45}$ which should be equal to $\Delta\mathcal{E}_{12}$ but is somewhat off.

MJS

3 2001 CAP exam VIII solution

a spherical current source, I, requires

$$\vec{J}(r) = \frac{I}{4\pi r^2}\,\hat{r}, \text{ in steady-state}$$

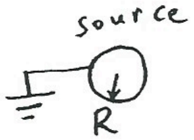
where J is the current density

$$\vec{J} = \sigma \vec{E} \quad \text{Ohm's Law}$$

where σ is the conductivity and $\vec{E}$ the electric field, σ behaves as $1/r^2$ in the problem

$$J(r) = \frac{I}{4\pi r^2} = \frac{\sigma_0}{r^2} E(r)$$

we see that $\vec{E} = \frac{I}{4\pi\sigma_0}\,\hat{r}$ is constant everywhere (outside the source)



$$\boxed{V = -\int_R^r \vec{E}\cdot d\vec{r} = -\frac{I}{4\pi\sigma_0}(r-R)}$$

$$\oint\!\!\oint \vec{E}\cdot d\vec{A} = \iiint_V \frac{\rho(r)}{\epsilon_0}\,dV$$

$$\frac{I}{4\pi\sigma_0}4\pi r^2 = \frac{4\pi}{\epsilon_0}\int \rho(r) r^2\,dr$$

differentiate with respect to r

$$\frac{2rI}{\sigma_0} = \frac{4\pi}{\epsilon_0}\rho(r) r^2$$

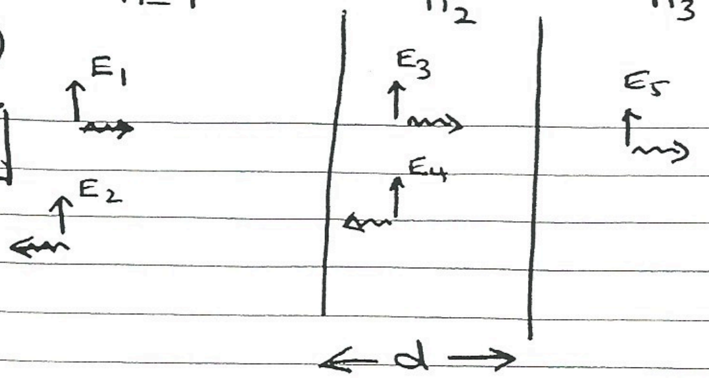
$$\rho(r) = \frac{\epsilon_0 I}{2\pi\sigma_0 r}$$

From Mark Chen.

#6

2001

from Ian Johnstone.



$H \sim E/\eta$ $\qquad \eta = \sqrt{\frac{\mu}{\epsilon}} \approx \sqrt{\frac{\mu_0}{\epsilon}}$

$\eta_0/\eta_2 = n_2$ $\qquad n_2 = \sqrt{\epsilon/\epsilon_0}$

$\eta_0/\eta_3 = n_3$

$$E_1 + E_2 = E_3 + E_4$$
$$E_1 - E_2 = n_2(E_3 - E_4)$$
$$E_3 e^{-i2\pi d/\lambda_2} + E_4 e^{+i2\pi d/\lambda_2} = E_5$$
$$E_3 e^{-i2\pi d/\lambda_2} - E_4 e^{+i2\pi d/\lambda_2} = n_3/n_2 \, E_5$$

when $d=0$

$$E_1 + E_2 = E_3 + E_4$$
$$E_1 - E_2 = n_2(E_3 - E_4)$$
$$E_3 + E_4 = E_5$$
$$E_3 - E_4 = n_3/n_2 \, E_5$$

$\therefore \quad E_1 + E_2 = E_5$

$E_1 - E_2 = n_3 E_5$

$\therefore \quad E_1 = \left(\frac{1+n_3}{2}\right) E_5, \qquad E_2 = \left(\frac{1-n_3}{2}\right) E_5$

$$\frac{E_2}{E_1} = \frac{(1-n_3)}{(1+n_3)} = \pm 0.5 \text{ from graph}$$

Obviously must be -0.5 for $n_3 > 1$

$$1 - n_3/1 + n_3 = -0.5$$
$$1 - n_3 = -0.5 - 0.5 n_3$$
$$0.5 n_3 = 1.5$$
$$\therefore n_3 = 3$$

When $d = 0.25\, d/\lambda_2$

$$E_1 + E_2 = E_3 + E_4$$
$$E_1 - E_2 = n_2(E_3 - E_4)$$
$$-i(E_3 - E_4) = E_5$$
$$-i(E_3 + E_4) = n_3/n_2\, E_5$$

$$\therefore E_1 + E_2 = -i\, n_3/n_2\, E_5$$
$$E_1 - E_2 = -i\, n_2\, E_5$$

$$\therefore E_1 = -i(n_2 + n_3/n_2)E_5$$
$$E_2 = -i(n_2 - n_3/n_2)E_5$$

$$E_2/E_1 = \frac{(n_2 - n_3/n_2)}{(n_2 + n_3/n_2)} = 0 \quad \text{from graph}$$

$$\therefore n_2 = n_3/n_2$$
$$n_2^2 = n_3$$
$$\therefore n_2 = \sqrt{3}$$

When $d = 0.5\, d/\lambda_2$

$$E_1 + E_2 = E_3 + E_4$$
$$E_1 - E_2 = \sqrt{3}(E_3 - E_4)$$
$$E_3 + E_4 = -E_5$$
$$E_3 - E_4 = -\sqrt{3}\, E_5$$

$$\therefore E_1 + E_2 = -E_5$$
$$E_1 - E_2 = -3E_5$$

$$\therefore E_1 = -4E_5$$
$$E_2 = 2E_5$$

$$\frac{E_2}{E_1} = -0.5 \quad \therefore \text{negative sign.}$$

b) From graph, η for $d_2/\lambda_2 = 1/8$ is about 0.35

$$\therefore E_2/E_1 \simeq -0.35$$

Standing wave $\quad |E_1| + |E_2| \simeq 1.35$

$$|E_1| - |E_2| \simeq 0.65$$

$\therefore$ Standing wave ratio $\simeq 2$

FROM IAN TURNER

CAP Exam 2001 Question IX

Radioactive decay law: rate of change of the population is proportional to the population

$$\frac{dN}{dt} = -\omega N$$

$$N(t) = N_0 e^{-\omega t}$$

where $N(t)$ = population at time t
N_0 = population at time $t=0$
ω = decay constant.

The half-life, $t_{1/2}$, is the time it takes for the population to reduce by a factor of 2, viz

$$\frac{1}{2} = e^{-\omega t_{1/2}} \Rightarrow t_{1/2} = \frac{\ln 2}{\omega}$$

The mean life, τ, is the average time that a nucleus is likely to survive before it decays:

$$\tau = \frac{\int_0^\infty t \left|{dN}/{dt}\right| dt}{\int_0^\infty \left|{dN}/{dt}\right| dt} \Rightarrow \tau = \frac{1}{\omega}.$$

The activity (counts per sec), $A = \omega N$. Thus

$$A(t) = A_0 e^{-\omega t} \qquad \text{where } A_0 = \omega N_0.$$

$$\ln[A(t)] = \ln[A_0] - \omega t$$

The usual method of measuring the half-life is to measure the activity, $A(t)$, as a function of time over a period that would span several half lifes. A plot of $\ln[A(t)]$ versus t should yield a straight line of slope $-\omega$ from which the $t_{1/2}$ is determined. The chemical composition of the source does not have to be measured.

This method clearly fails for sources of very long half lifes as there is no appreciable change in the activity over the lifetime of the experiment. Instead the activity itself is measured, $A = \omega N$, and then the number f atoms, N, has to be determined (for example, by weighing the sample hose chemical composition is accurately known).

The activity is measured by counting the electrons in the decay

$$^{87}Rb \rightarrow {}^{87}Sr + e^- + \bar{\nu}_e$$

However the electrons have very low energy, $E_e \leq 0.272$ MeV, thus there is a real possibility that the electron in collisions with other atoms will get topped in the source and not emerge to be counted. To minimize this ffect, very thin sources are used. This reduces the activity and hence the ount rate -- making the experiment more difficult. Detectors capable of easuring low energy electrons are often difficult to calibrate -- another roblem for the experiment.

3. Let t_0 = the time at which the meteorites are formed

t_1 = today's time

$t_1 - t_0$ = age of the meteorites

$N_1(t)$ = number of atoms of ^{87}Rb at time t

$N_2(t)$ = number of atoms of ^{87}Sr at time t

N_3 = number of atoms of ^{86}Sr

Since ^{86}Sr is not a daughter product of any long-lived radioactive nucleus, and is stable, the population N_3 is constant with time.

The population N_1 varies with time

$$N_1(t_1) = N_1(t_0)\, e^{-\omega(t_1 - t_0)}$$

However, the loss in the number of atoms of ^{87}Rb is matched by the gain in the number of atoms of ^{87}Sr. That is

$$N_1(t_1) + N_2(t_1) = N_1(t_0) + N_2(t_0)$$

Divide all through by N_3

$$\frac{N_2(t_1)}{N_3} = \frac{N_1(t_0)}{N_3} - \frac{N_1(t_1)}{N_3} + \frac{N_2(t_0)}{N_3}$$

$$= \frac{N_1(t_1)}{N_3} e^{\omega(t_1-t_0)} - \frac{N_1(t_1)}{N_3} + \frac{N_2(t_0)}{N_3}$$

$$\frac{N_2(t_1)}{N_3} = \frac{N_1(t_1)}{N_3}\left[e^{\omega(t_1-t_0)} - 1\right] + \frac{N_2(t_0)}{N_3} \quad \text{— (1)}$$

Assuming that the primordial ratio $^{87}Sr/^{86}Sr = N_2(t_0)/N_3$ is fixed, then eq. (1) takes the form of

$$y = m x + c$$

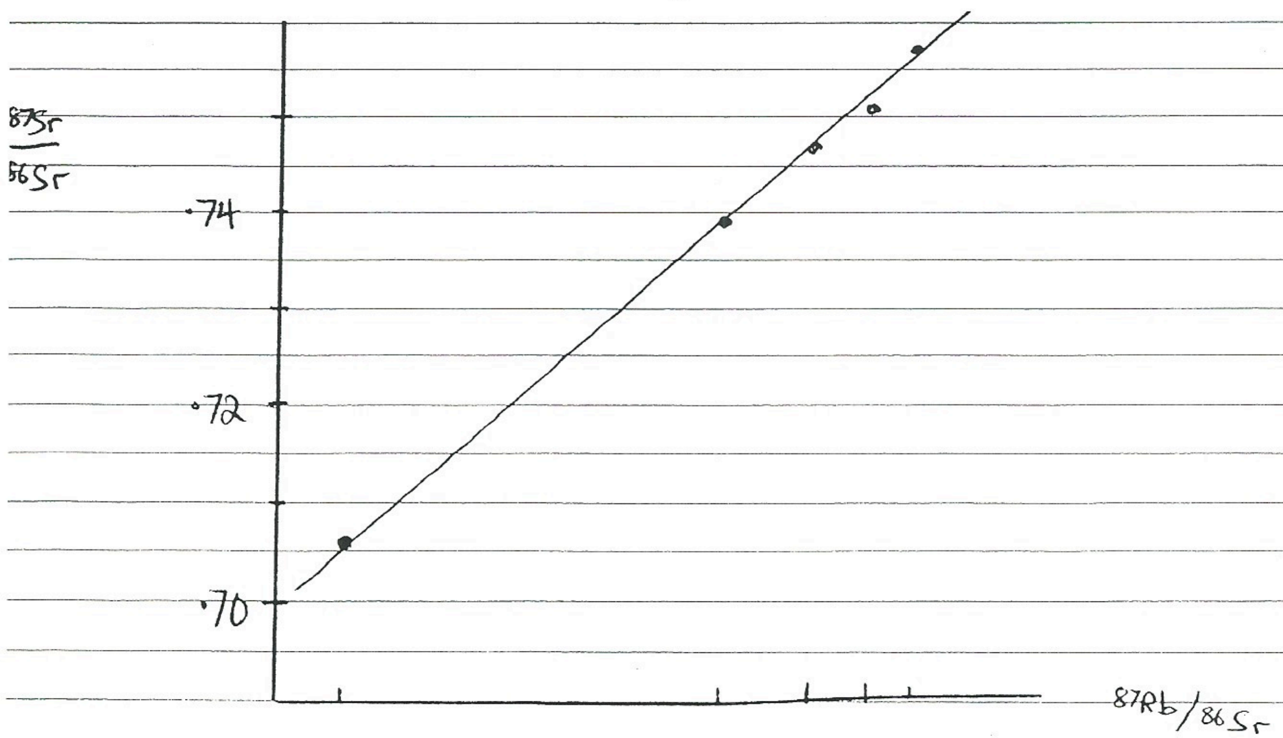
$y = \frac{^{87}Sr}{^{86}Sr} = \frac{N_2(t_1)}{N_3}$ at today's time

$x = \frac{^{87}Rb}{^{86}Sr} = \frac{N_1(t_1)}{N_3}$ at today's time

$m = \left[e^{\omega(t_1-t_0)} - 1\right]$

$c = N_2(t_0)/N_3 = {^{87}Sr}/{^{86}Sr}$ at $t=t_0$

Plotting the given data



The sloke

$$m = \frac{0.757-0.706}{0.86-0.09} = \frac{0.051}{0.77} = 0.0662$$

$$e^{w(t_1-t_0)} - 1 = 0.0662$$

$$e^{w(t_1-t_0)} = 1.0662$$

$$t_1-t_0 = \frac{1}{w} \ln(1.0662)$$

$$= \frac{t_{1/2}}{\ln 2} \ln(1.0662) \qquad t_{1/2} = 4.7\times10^{10}\ y$$

$$\boxed{\text{Common age of samples} = 4.35\times10^{9}\ \text{yrs.}}$$

2001. CAP EXAM From Larry Widrow

10) This problem deals with the propagation of electromagnetic waves in a plasma, that is, a region of free electrons. The cut-off frequency referred to in the problem is the plasma frequency, $\omega_p^2 = \frac{Ne^2}{\epsilon_0 m_e}$

which we can derive as follows

Propagation of E;M waves are governed by Maxwell's equations

$$\vec{\nabla} \cdot \vec{B} = 0. \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{B} - \mu_0 \frac{\partial \vec{D}}{\partial t} = 0$$

$$\text{or} \quad \vec{\nabla} \times \vec{B} - \frac{1}{\epsilon_0 c^2} \frac{\partial \vec{D}}{\partial t} = 0$$

We need a relation between $\vec{D}$ + $\vec{E}$

$$\vec{D} = \vec{P} + \epsilon_0 \vec{E}$$

$\vec{P}$ = dipole moment / volume

or single electron $\qquad m\ddot{\vec{x}} = -e\vec{E}$

+ dipole moment / electron $\qquad \vec{p} = -e\vec{x}$

for harmonic wave $\sim \vec{E} \sim e^{-i\omega t}$ we have

$$-m\omega^2 \vec{x} = -e\vec{E} \Rightarrow p = -\frac{e^2}{m\omega^2}\vec{E}$$

Thus we have $\qquad \vec{D} = \epsilon_0\left(1 - \frac{Ne^2}{m\omega^2\epsilon_0}\right)\vec{E}$

$$= \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E}$$

where $\omega_p^2 = \frac{Ne^2}{m\epsilon_0}$ $\qquad = \epsilon(\omega)\vec{E}$

$$\epsilon(\omega) = \epsilon_0\left(1 - \omega_p^2/\omega^2\right)$$

Combining $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

and $\vec{\nabla} \times \vec{B} - \frac{\epsilon(\omega)}{\epsilon_0} \frac{1}{c^2} \frac{\partial E}{\partial t} = 0$

curl of 2nd equation gives

$$- \nabla^2 \vec{B} + \frac{\epsilon(\omega)}{\epsilon_0} \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

for $\vec{B} \sim e^{i(kz - \omega t)}$ we get dispersion relation

$$k^2 - \frac{\omega^2 + \omega_p^2}{c^2} = 0$$

$$\omega = (c^2 k^2 + \omega_p^2)^{1/2}$$

$$v_{phase} = \frac{\omega}{k} = c\left(1 + \frac{\omega_p^2}{c^2 k^2}\right)^{1/2} > c$$

$$v_{group} = \frac{d\omega}{dk} = c\left(1 + \frac{\omega_p^2}{c^2 k^2}\right)^{-1/2} < c$$

a) Calculate $f_p = 2\pi\omega_p$

$$= 2\pi \left(\frac{Ne^2}{\epsilon_o m_e} \right)^{1/2}$$

$N = 2 \times 10^{+9} \ m^{-3}$

$e = 1.6 \times 10^{-19} \ C$

$\epsilon_o = 8.85 \times 10^{-12} \ \frac{C^2}{Nm^2}$

$m_e = 9.109 \times 10^{-31} \ kg$

$$f_p = 2\pi \left(\frac{2 \times 10^{9} \ \cancel{m^{3}} \ (1.6 \times 10^{-19})^2 \ \cancel{C^2} \ \cancel{kg} \ \cancel{m^3}/s^2}{8.85 \times 10^{-12} \ \cancel{C^2} \ 9.11 \times 10^{-31} \ \cancel{kg}} \right)^{1/2}$$

$= 1.58 \times 10^{7} \ Hz$ or $15.8 \ MHz$

b) $\Delta t = t_1 - t_2$

$$= \frac{d}{v_1} - \frac{d}{v_2}$$

$$= \frac{d}{c}\left(\left(1 + \frac{f_p^2}{f_1^2}\right)^{1/2} - \left(1 + \frac{f_p^2}{f_2^2}\right)^{1/2}\right)$$

sing Taylor exp. $= \frac{d}{c} \times \frac{1}{2} \times f_p^2 \left(\frac{1}{f_1^2} - \frac{1}{f_2^2}\right)$

$$= 0.8 \text{ years} \times \left(\left(\frac{15.8}{15 \times 10^3}\right)^2 - \left(\frac{15.8}{20 \times 10^3}\right)^2\right)$$

$$= 12.3 \text{ s}$$

c) $\Delta \epsilon = \frac{\epsilon(\omega) - \epsilon_0}{\epsilon_0} \simeq - \frac{f_p^2}{f_1^2}$

$$\simeq -\left(\frac{1.58 \times 10}{1.5 \times 10^4}\right)^2 = -1.11 \times 10^{-6}$$