2005 Canadian Association of Physicists Prize Examination

Tuesday, February 8, 2005

Duration: 3 hours

Instructions:

You are permitted to use calculators for the exam.

Each question is to be answered in a separate exam booklet. The number of the question, the name of the candidate, and the name of the university/department should be clearly indicated on the first page of each booklet.

Attempt as many questions as possible, in whole or in part. It is not likely that you will be able to complete all questions, so work primarily on those questions you feel most able to answer.

Each question holds the same value.

Good luck! We hope you enjoy the experience.

The completed examination booklets should be sent by Department Chairpersons to:

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This year's exam is a collaboration between UPEI (Doug Dahn, Sheldon Opps), Mount Allison University (Mohammad Ahmady), Université de Moncton (Jean Desforges), Acadia University (Svetlana Barkanova), St. Mary's University (Joe Hahn), and Saint Francis Xavier University (Carl Adams, Robert Wickham).

1. A thin rod of length L has a linear mass density that varies as $\lambda(x) = \lambda_0 x/L$, where x is the distance from one end of the rod, and λ_0 is a constant. The rod is suspended from a pivot point at its light end, and is also subject to a constant gravitational acceleration g.

   (a) How far is the rod's centre of mass from its pivot point?

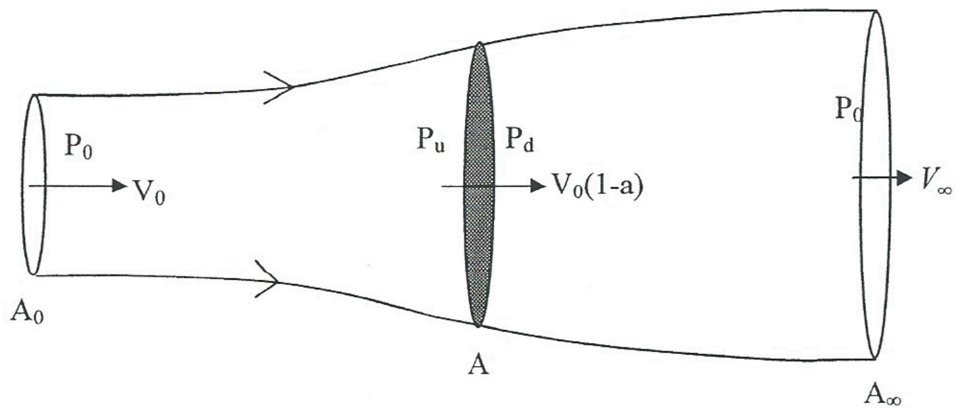
   (b) What is the rod's moment of inertia about its pivot point?

   (c) Let θ be the rod's angular displacement from a vertical orientation. What is the torque that gravity exerts on the rod about its pivot point?

   (d) What is the rod's natural frequency of small oscillations?

2. (a) Air of density ρ flows at a uniform wind speed v_0 through an area A, which is normal to the wind direction. At what rate is kinetic energy transported through A?

   (b) We now want to extract some of this energy by building a windmill, as shown in the figure below. The blades of the windmill sweep out area A. Far upstream of the windmill, the air velocity is the undisturbed wind speed v_0. The velocity as the air passes through the windmill is $v_0(1-a)$, and far downstream it is v_∞. The figure shows streamlines of air flow, bounding a stream tube of area A_0 far upstream, and A_∞ far downstream. Far away from the windmill in either direction the pressure is atmospheric pressure P_0. Just upstream of the windmill the pressure is P_u, and just downstream it is P_d.



   Assuming that the air is incompressible, and that it behaves as an ideal fluid except where it interacts with the windmill, use Bernoulli's equation to find the pressure discontinuity $\Delta P = P_u - P_d$ as a function of ρ, v_0, and v_∞.

(c) By considering the flow of momentum into and out of the system, show that the force F exerted on the windmill is given by $F = \rho\left(A_0 v_0^2 - A_\infty v_\infty^2\right)$.
(d) We also know that $F = A\, \Delta P$. Combining this with the result from (c), and using the fact that mass is conserved, you should be able to eliminate v_∞ from the equation for F. Then, find the power extracted from the wind by the windmill.

(e) What value of the parameter a maximizes the power extracted from the wind? For this optimum case, what percentage of the power found in part (a) is extracted?

3. A mass m of water at T_1 is isobarically and adiabatically mixed with an equal mass of water at T_2. Show that the entropy change of the universe is

$$2mC_p \ln\left(\frac{(T_1+T_2)/2}{\sqrt{T_1T_2}}\right),$$

and prove that this is positive. C_p is the specific heat at constant pressure for water, which you can assume is independent of temperature. Hint: How do you prove that the final temperature is

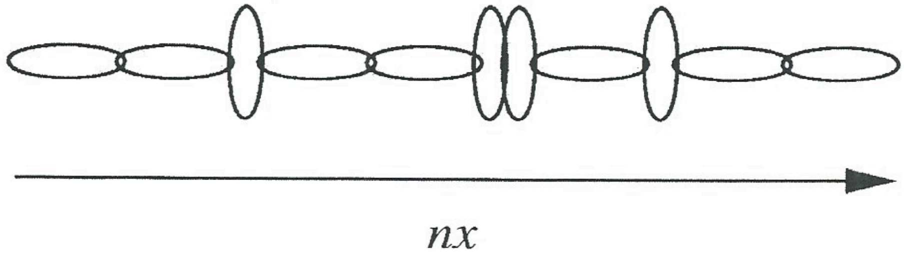
$$T_F = \frac{1}{2}(T_1+T_2)\ ?$$

4. Consider a one-dimensional chain consisting of $n >> 1$ link segments as illustrated in the figure below. Let the length of each link be a when the long dimension of the link is parallel to the chain and zero when the link is vertical (i.e., the long dimension is normal to the chain direction). Each segment has two distinct states, a horizontal orientation and a vertical orientation. The distance between the ends of the chain is nx.

(a) Find the entropy of the chain as a function of x.

(b) Obtain a relation between the temperature T of the chain and the tension F which is necessary to maintain the distance nx, assuming the joints turn freely.

(c) Under which conditions does your answer lead to Hooke's law?



5. A particle of mass m is subject to a central force that varies as $F(r) = -kr^2$, where k is a positive constant and r is the distance from the force origin.

   (a) What is the system's Lagrangian?

   (b) What are the Lagrange equations of motion?

   (c) Identify two conserved quantities.

   (d) Find the frequency of small radial oscillations about a circular orbit of radius $r = b$.

6. (a) A muon (a particle with charge –e and a mass equal to 207 times the mass of the electron) is captured by a deuteron to form a muonic atom. Find the energy of the ground state and the first excited state.

   (b) A train moves with velocity v relative to the ground. A bird, flying in the same direction along the railway, moves with velocity v relative to the train. A plane, moving in the same direction as well, has velocity v relative to the bird. According to Newton, what is the velocity of the plane relative to the ground? According to Einstein, what is the velocity of the plane relative to the ground?

7. (a) Compute the commutator $[e^{ia\hat{p}/\hbar}, \hat{x}]$.

   (b) It is known that two operators $\hat{A}$ and $\hat{B}$ commute as $\hat{A}\hat{B} - \hat{B}\hat{A} = 1$. Find $\hat{A}\hat{B}^2 - \hat{B}^2\hat{A}$.

8. The half infinite magnetic slab.

   (a) First, consider an infinite sheet of current in the x-y plane, with current flowing in the x-direction. A 2-dimensional current density is sometimes denoted by **K**. In this case,

$$\vec{K} = K_0\hat{i}$$

   for z = 0. There are no other currents in the problem, and the current flow is steady. This is also a *free* current.

   Make some arguments using as applicable the Biot-Savart law, Gauss' law for magnetism, and the symmetry of the problem to make some conclusions about the form of the magnetic field **B**.

   Calculate the magnetic field above and below the plane using Ampére's Law. (Hint: the symmetry of the problem suggests that $\vec{B}(x,y,z) = -\vec{B}(x,y,-z)$.)

(b) Now suppose that the region below the x-y plane is filled with a magnetic material with magnetic susceptibility χ_m. In response to the infinite current sheet a bound current $\mathbf{K}_b$ will be formed at the x-y plane. The method to solve this problem is to use the auxiliary field **H**, which obeys Ampére's Law for free currents, $\nabla \times \vec{H} = \vec{J}_f$, make sure Gauss' Law for magnetism $\nabla \cdot \vec{B} = 0$ is satisfied, while maintaining the relationship $\vec{B} = \mu \vec{H} = (1 + \chi_m)\mu_0 \vec{H}$. The free currents are unchanged from part (a). Solve this problem for **B** and **H** in the different regions, knowing that $\vec{B}(x,y,z) = -\vec{B}(x,y,-z)$ still holds.

9. For a particle of mass m in an infinite potential well the energy eigenfunctions and eigenvalues are given as

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}, \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \qquad n = 1,2,3,\ldots,$$

where a is the width of the well.

(a) What is the most general form of the wavefunction $\psi(x,t)$ if energy measurement at t = 0 finds the particle either in the ground state or the first excited state with equal probabilities?

(b) Use the wavefunction in part (a) to calculate $\langle x \rangle$ at time t. What is the maximum value of $\langle x \rangle$?

(c) Find the particular wavefunction $\psi(x,t)$ for which $\langle x \rangle$ has its maximum at t = 0.

(d) Calculate the expectation value of the momentum $\langle p \rangle$ for the wavefunction in part (c). What is the maximum of $\langle p \rangle$?

10. The sound produced by a guitar string plucked at t = 0 can be modeled as a damped oscillation as follows:

$$f(t) = 0, \quad t < 0,$$
$$f(t) = A\, e^{-at} \cos \omega_1 t, \quad t > 0.$$

(a) Find the Fourier transform F(ω) of f(t). You should find that it is proportional to

$$A\left[\frac{1}{a - i(\omega + \omega_1)} + \frac{1}{a - i(\omega - \omega_1)}\right].$$

(b) If you hear a clear musical note which gradually fades away, what does this say about the relative sizes of ω_1 and a? In this case, one can show that one of the terms

Jan Towner

2005 CAP Exam

Qu. 1. Mass of rod

$$M = \int_0^L \lambda(x)\,dx = \frac{\lambda_0}{L}\int_0^L x\,dx = \frac{\lambda_0}{L}\frac{L^2}{2}$$

$$= \frac{1}{2}\lambda_0 L$$

a) Centre of mass of the rod

$$X = \frac{1}{M}\int_0^L \lambda(x)\,x\,dx = \frac{\lambda_0}{ML}\int_0^L x^2 dx = \frac{\lambda_0}{ML}\frac{L^3}{3}$$

$$= \frac{\lambda_0}{\frac{1}{2}\lambda_0 L}\frac{L^2}{3}$$

$$= \frac{2}{3}L$$

b) Moment of inertia about pivot point

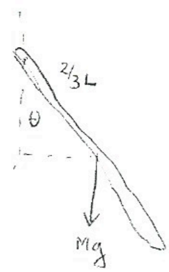
$$I = \int_0^L \lambda(x)\,x^2 dx = \frac{\lambda_0}{L}\int_0^L x^3 dx = \frac{\lambda_0}{L}\frac{L^4}{4}$$

$$= \frac{1}{4}\lambda_0 L^3$$

$$= \frac{1}{2}ML^2$$

c) $$\text{Torque} = Mg \times \frac{2}{3}L\sin\theta$$

$$= \frac{1}{3}\lambda_0 g L^2 \sin\theta$$



d) Kinetic Energy = Translation of centre of mass + Rotation about centre of mass

$= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2} I_g \dot{\theta}^2$

$X = \frac{2}{3} L \sin\theta$

$Y = \frac{2}{3} L \cos\theta$

I_g = mom. of inertia about centre of mass

= Translation of pivot + Rotation about pivot

$= 0 + \frac{1}{2} I \dot{\theta}^2$

I = mom. of inertia about pivot

$= \frac{1}{2}\left(\frac{1}{2}ML^2\right)\dot{\theta}^2$

Potential Energy $= -MgY = -Mg\frac{2}{3}L\cos\theta$

Lagrangian $L = \frac{1}{4}ML^2\dot{\theta}^2 + Mg\frac{2}{3}L\cos\theta$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ML^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2}ML^2\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = -\frac{2}{3}MgL\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{1}{2}ML^2\ddot{\theta} + \frac{2}{3}MgL\sin\theta = 0$

$\ddot{\theta} = -\frac{4}{3}\frac{g}{L}\sin\theta$

For small angles

$\ddot{\theta} = -\frac{4}{3}\frac{g}{L}\theta = -\omega^2\theta$

$\omega = \sqrt{\frac{4}{3}\frac{g}{L}}$ = frequency of small oscillations

CAP 2005

DRT

2 (a) In Δt, mass Δm crosses A, carrying $\Delta T = \frac{1}{2}\Delta m V_0^2$

Rate of K.E. transport $= \frac{dT}{dt} = \frac{1}{2}\frac{dm}{dt}V_0^2$

$$\frac{dm}{dt} = \dot{m} = \rho\dot{V}$$
$$= \rho A\dot{x} = \rho A V_0$$
$$\therefore \dot{T} = \frac{1}{2}\rho A V_0^3$$

(b) Bernoulli : $P + \frac{1}{2}\rho v^2 = \text{const}$ (no potential energy, incompressible)

Between A_0 and $A(u)$: $P_0 + \frac{1}{2}\rho V_0^2 = P_u + \frac{1}{2}\rho V_0^2(1-a)^2$

Between A_∞ and $A(d)$: $P_0 + \frac{1}{2}\rho V_\infty^2 = P_d + \frac{1}{2}\rho V_0^2(1-a^2)$

$$\therefore P_u - P_d = \frac{1}{2}\rho(V_0^2 - V_\infty^2)$$

(c) At A_0, $\dot{p} = \dot{m}V_0 = \rho A_0 V_0^2$

At A_∞ : $\dot{p} = \dot{m}V_\infty = \rho A_\infty V_\infty^2$

Net $\dot{p}$ = Force on air $= \rho(A_\infty V_\infty^2 - A_0 V_0^2)$

= − Force on windmill

$$\therefore F = \rho(A_0 V_0^2 - A_\infty V_\infty^2)$$

(d) Assume $F = A\Delta P = \frac{1}{2}\rho A(V_0^2 - V_\infty^2) = \rho(A_0V_0^2 - A_\infty V_\infty^2)$

Also $\rho A_0 V_0 = \rho A_\infty V_\infty$

$$\left\{\begin{array}{l} V_\infty = V_0(A_0/A_\infty) \\ \rho(A_0V_0) = \rho A V_0(1-a) \end{array}\right.$$

Can use these to eliminate V_∞, A_∞, A in F

$$A = A_0/(1-a)$$

$$F = \frac{1}{2}\rho\frac{A_0}{(1-a)}\left(V_0^2 - V_0^2\left(\frac{A_0}{A_\infty}\right)^2\right) = \rho\left(A_0V_0^2 - A_\infty V_0^2\frac{A_0^2}{A_\infty^2}\right)$$
$$= \rho A_0V_0^2\left(1 - \frac{A_0}{A_\infty}\right)$$

$$\frac{1}{2}\rho\frac{A_0}{1-a}V_0^2\left(1+\frac{A_0}{A_\infty}\right)\left(1-\frac{A_0}{A_\infty}\right) = \rho A_0V_0^2\left(1-\frac{A_0}{A_\infty}\right)$$

$$1 + \frac{A_0}{A_\infty} = 2(1-a) = 2-2a$$

$$\frac{A_0}{A_\infty} = 1-2a$$

$$A_\infty = \frac{A_0}{1-2a}$$

So $F = \rho(A_0 V_0^2 - A_\infty V_\infty^2)$

$$= \rho\left(A_0 V_0^2 - \left(\frac{A_0}{1-2a}\right)\frac{V_0^2 A_0^2 (1-2a)^2}{A_0^2}\right)$$

$$= \rho A_0 V_0^2 \left[1 - [1-2a]\right] = 2a\rho A_0 V_0^2$$

In general $P = \vec{F} \cdot \vec{V}$ and $V = V_0(1-a)$

(some assumptions required?)

$$P = 2a(1-a)\rho A_0 V_0^3$$

(e) P is maximum for $\frac{dP}{da} = 0$

$$(1-a) - a = 0$$

$$a = 1/2$$

$$P_{max} = \frac{1}{2}\rho A_0 V_0^3 \qquad (\text{Note } A_\infty \to \infty \text{ for } a = \tfrac{1}{2})$$

From (a) power "available" $= \frac{1}{2}\rho A V_0^3$

$$A = \frac{A_0}{1-a} = 2A_0$$

$$\to \frac{1}{\cancel{2}} \times \cancel{2} \rho A_0 V_0^3$$

So 50% of power is extracted

An alternative analysis is to use the energy balance equation at A_0 and A_∞

$$\dot{Q} - \dot{W}_{shaft} = \dot{m}(\Delta h + \Delta ke + \Delta pe)$$

$\dot{Q}$ positive in
$\dot{W}_{shaft}$ positive out

Ideality, incompressibility implies $\Delta h = 0$

Maximum $\dot{W}_{shaft}$ when $\dot{Q} = 0$ $\quad (\dot{Q} \leq 0)$

So $\dot{W}_{max} = -\frac{\dot{m}}{2}(V_\infty^2 - V_0^2)$ $\qquad V_\infty = \left(\frac{A_0}{A_\infty}\right) V_0$ as before

$$= \frac{\dot{m}}{2} V_0^2 \left(1 - \left(\frac{A_0}{A_\infty}\right)^2\right)$$

$$= \frac{1}{2}\rho A_0 V_0^3 \left(1 - (A_0/A_\infty)^2\right)$$

This is maximum for $A_\infty \to \infty \to \frac{1}{2}\rho A_0 V_0^3$

3. Choose $T_1 > T_2$. Let final temperature be T_f

If water is incompressible $C_p = C_v$

Mass m at T_1 loses energy $\Delta U_1 = mC_p(T_1 - T_f)$

" " " T_2 gains energy $\Delta U_2 = mC_p(T_f - T_2)$

First law $\Delta U_1 = \Delta U_2$

$$T_1 - T_f = T_f - T_2$$

$$T_f = \frac{T_1 + T_2}{2}$$

$T dS = dU + P dV$ $\quad$ $dV = 0$ (incompressible)

$$dS = \frac{dU}{T} = \frac{mC_p\,dT}{T}$$

For mass at T_1, loss of entropy is

$$\Delta S_1 = mC_p \int_{T_1}^{T_f} \frac{dT}{T} = mC_p \ln\left(\frac{T_f}{T_1}\right)$$

For mass at T_2, gain of entropy is

$$\Delta S_2 = mC_p \int^{T_f} \frac{dT}{T} = mC_p \ln \frac{T_f}{T_2}$$

Net gain of entropy is $\Delta S_2 + \Delta S_1$

$$= mC_p \ln \frac{T_f}{T_2} + mC_p \ln \frac{T_f}{T_1}$$

$$= mC_p \ln \frac{T_f^{\,2}}{T_2 T_1}$$

$$= 2mC_p \ln\left(\frac{T_f}{\sqrt{T_1 T_2}}\right) = 2mC_p \ln\left(\frac{(T_1+T_2)/2}{\sqrt{T_1 T_2}}\right)$$

Positive if $\dfrac{T_1 + T_2}{2} > \sqrt{T_1 T_2}$

square: $\dfrac{T_1^2}{4} + T_1 T_2 + \dfrac{T_2^2}{4} \overset{?}{>} T_1 T_2$

True

Steve Hughes

From shughes@physics.queensu.ca Thu Oct 6 13:18:01 2005

Hi Malcolm,

I had a quick go at that linear chain problem last night but I do not have my stat mech book on hand (it's in Vancouver!).

a) S = kb ln P(l=nx=ma), [m chains facing to the horizontal]

where P(l) is obtained by applying some stats. Specifically the probably of horizontal or vertical numbers in a large n chain can be simplified with Stirling formula to get a Gaussian distribution. Normalizing in 1d then gives S, with the ln cancelling the exp from the Gaussian. I get:

S(nx) = - C K1 kb (2(nx)-na)^2

This is actually a bit tricky to do from scratch, so maybe there is an easier way.

(K1 by the way is 2/(sqrt(2pi)(na)^1.5), but I may have that normalized incorrectly, but it does not really matter.)

b) Next, the force strain is F = -T dS/dl (by definition I think and no doubt neglecting something), so by plugging in we get:

F(T) = - 2 K1 kb T(2(nx)-na)

c) Hooke's law is indeed obeyed only when nx > na/2, which is as expected I guess, i.e. when stretching from equilibrium like an elastic band. Note on stretching further, this must break down which tells us that the model is not very good (or that my attempt at the problem is not very good!)

However, I might very well be barking up the wrong tree as my stat mech is very rusty and Bob Gooding will no doubt have a much more elegant solution.

Cheers,

- Steve

[A LUNCH TIME ATTEMPT – BUT I CANNOT FIND MY (QUAN) STAT MECH BOOK]

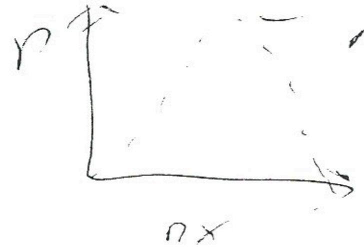
1/ Probability of m horizontal chains. S.A

(a) $$P_{n,m} = \frac{2^n n!}{m!(n-m)!}$$

and ENTROPY $S = k_B \ln[\bar{P}^{(nx)}_{n,m}]$

For n and $m \gg 1$, and $m \simeq n/2$ one can expand using Stirling approximation to factorials [$n! \simeq \sqrt{2\pi}\, e^{-n} n^{n+1/2}$] to get:

$$P_{n,m} \simeq C \exp - \frac{(2m-n)^2}{2n}$$



max when $n = 2m$ as expected.

Converting to $l = nx = ma$

$$P(l) = C \exp\left[-\frac{(2nx - na)^2}{2na}\right]$$

$\Rightarrow$ on normalizing

$$\bar{P}(l) = \frac{2}{\sqrt{2\pi a n}} \exp - \frac{(2l - na)^2}{2na}$$

$\Rightarrow S = k_B \ln[\ \ ]$

$$\boxed{S(nx) = -\frac{k_B [2nx - na]^2}{\sqrt{2\pi}\,(na)^{3/2}}} \quad (1)$$

2/

(b) $F = -T \frac{\partial S}{\partial L}$ = force drawn to keep $L_0 = nx_0$

$$\Rightarrow F(T) = -\frac{T k_B 4 [(2nx) - na]}{\sqrt{2\pi}\,(na)^{3/2}} \qquad (2)$$

(c) Could calculate $\langle L \rangle = \langle nx \rangle$ subject to an extra force, but no need.

(2) obviously obeys Hooke's law as expected when

$$2nx - na > 0$$

$$\Rightarrow nx > \frac{na}{2}$$

Thus when stretched from equilibrium like an elastic band, though probably a very crude model as would also break down if x is too large!

Jan Townend

2005 CAP Exam

Qu. 5 a) Conservative force $F = -\frac{dV}{dr} = -kr^2$

Potential energy $V = \int kr^2 dr = \frac{k}{3}r^3$

Motion under central forces is confined to a plane, hence use plane polar coordinates

Kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{3}r^3$

b) There are two Lagrange equations of motion

i) Equation in r: $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - kr^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + kr^2 = 0}$$

ii) Equation in θ: $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\boxed{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0}$$

But since $\partial L/\partial \theta = 0$, we have $\frac{d}{dt}(\partial L/\partial \dot{\theta}) = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}$$

c) Constants of motion

i) Observe that θ = cyclic coordinate, $\partial L/\partial\theta = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = 0$$

$$p_\theta \equiv \frac{\partial L}{\partial\dot\theta} = \text{constant}$$

$$\boxed{p_\theta = mr^2\dot\theta}$$

ii) The Lagrangian does not depend explicitly on time, therefore the Hamiltonian $H = T+V$ is a constant of motion, the total energy

$$\boxed{H = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{k}{3}r^3 = \text{constant}}$$

d) Solve the radial equation. First replace $\dot\theta$ by p_θ/mr^2

$$m\ddot r - mr\left(\frac{p_\theta}{mr^2}\right)^2 + kr^2 = 0$$

$$m\ddot r - \frac{p_\theta^2}{mr^3} + kr^2 = 0$$

Solve by writing $r = b+s$, where s is a variable that is considered small for "small radial oscillations"

$$\dot r = \dot s$$
$$\ddot r = \ddot s$$

$$m\ddot s - \frac{p_\theta^2}{m(b+s)^3} + k(b+s)^2 = 0$$

$$(b+s)^{-3} = b^{-3}(1+s/b)^{-3} = b^{-3}(1-3s/b\ \cdots)$$

$$(b+s)^2 = b^2(1+s/b)^2 = b^2(1+2s/b\cdots)$$ to first order in s

$$m\ddot{s} - \frac{p_\theta^2}{mb^3}\left(1 - \frac{3s}{b} \cdots\right) + kb^2\left(1 + 2s/_b \cdots\right) = 0$$

$$m\ddot{s} - \frac{p_\theta^2}{mb^3} + \frac{3p_\theta^2}{mb^4}s + kb^2 + 2kbs = 0.$$

$$m\ddot{s} = -\left(2kb + \frac{3p_\theta^2}{mb^4}\right)s + \frac{p_\theta^2}{mb^3} - kb^2$$

$$\ddot{s} = -\left(\frac{2kb}{m} + \frac{3p_\theta^2}{m^2b^4}\right)s + \frac{p_\theta^2}{m^2b^3} - \frac{kb^2}{m}$$

This equation is of the form

$$\ddot{s} = -\omega^2 s + \text{constant}$$

which has an oscillating solution of frequency

$$\omega = \left(\frac{2kb}{m} + \frac{3p_\theta^2}{m^2b^4}\right)^{1/2}$$

Jon Towner

2005 CAP Exam

7 a) First we need the commutator $[\hat{p}^n, \hat{x}]$.

Build up recursively: $[\hat{p}, \hat{x}] = -i\hbar$

$$[\hat{p}^2, \hat{x}] = \hat{p}\,[p, x] + [p, x]\,x$$
$$= -i\hbar\hat{p} - i\hbar\hat{p}$$
$$= -2i\hbar\hat{p}$$

$$[\hat{p}^3, \hat{x}] = \hat{p}\,[\hat{p}^2, \hat{x}] + [p, \hat{x}]\,\hat{p}^2$$
$$= -2i\hbar\hat{p}^2 - i\hbar\hat{p}^2$$
$$= -3i\hbar\hat{p}^2$$
$$\vdots$$
$$[\hat{p}^n, \hat{x}] = -ni\hbar\,\hat{p}^{n-1}$$

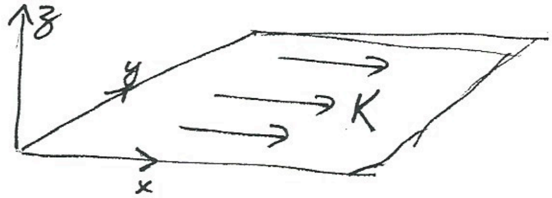
Write $$e^{+ia\hat{p}/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{+ia}{\hbar}\right)^n \hat{p}^n$$

$$[e^{ia\hat{p}/\hbar}, \hat{x}] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n [\hat{p}^n, \hat{x}]$$
$$= -i\hbar \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{ia}{\hbar}\right)^n n\,\hat{p}^{n-1}$$
$$= -i\hbar\,\frac{ia}{\hbar} \sum_{n=0}^{\infty} \frac{1}{(n-1)!}\left(\frac{ia}{\hbar}\right)^{n-1} \hat{p}^{n-1}$$
$$= a\,e^{ia\hat{p}/\hbar}$$

b) Given $[\hat{A}, \hat{B}] = 1$

$$[A, \hat{B}^2] = [\hat{A}, \hat{B}]\hat{B} + \hat{B}[\hat{A}, \hat{B}]$$
$$= \hat{B} + \hat{B}$$
$$= 2\hat{B}$$

8a)



Biot-Savart tells us

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \vec{r}}{r^3}$$

From symmetry, we ~~can find~~ have $\vec{B}(0,0,z_0) = \vec{B}(x,y,z_0)$

$d\vec{l} = dx\ \hat{i}$

$\vec{l} = (x, y, 0)$ = current point

$\vec{r} = (0,0,z_0) - (x,y,0) = (-x, -y, z_0)$

$I = \int K\, dy$

so $$\vec{B} = \frac{\mu_0 K}{4\pi} \int dy\, dx \ \frac{\hat{i} \times (-x, -y, z_0)}{(x^2+y^2+z_0^2)^{3/2}}$$

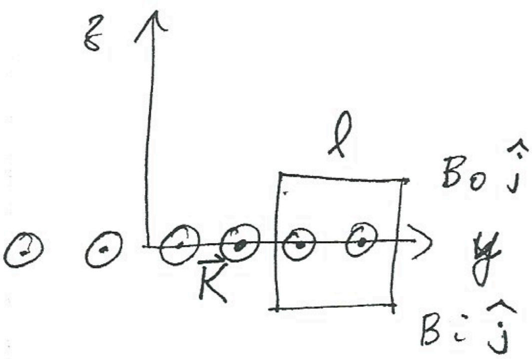
$$= \frac{\mu_0 K}{4\pi} \int dy\, dx \ \frac{(0, -z_0 \cancel{dx}, -y \cancel{dx})}{(x^2+y^2+z_0^2)^{3/2}}$$

The z component is zero, since the ~~integral~~ function is odd. Also $B(x,y,z_0) = -B(x,y,-z_0)$, by inspection

So $\vec{B}(x,y,z_0) = B_y(x,y,z_0) \equiv B_0\, \hat{j}$

Also B is independent of z. To see this, ~~see~~ use $\vec{\nabla} \times \vec{B} = 0$ around a small loop above the plane.

8a) To find $\vec{B}$,



$$\vec{\nabla} \times \vec{B} = \mu_0 I$$

$$\oint \vec{\nabla} \times \vec{B} \, dA = -\mu_0 K l$$

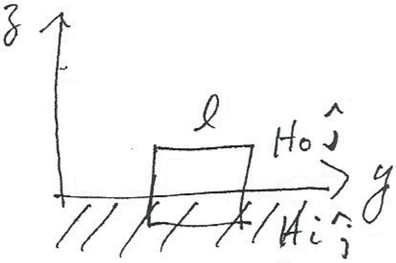
$$B_0 l - B_i l = -\mu_0 K l$$

$$B_0 - B_i = -\mu_0 K$$

$$2B_0 = -\mu_0 K$$

$$\boxed{B_0 = -\frac{\mu_0 K}{2}}$$

b) $\vec{\nabla} \times \vec{H} = J_f$.



$\mu = (1 + \chi_m)\mu_0$

$$H_0 l - H_i l = K_f l$$

$$H_0 - H_i = K_f$$

$$B_0 = -B_i$$

$$\mu_0 H_0 = - \mu H_0$$

$$\Rightarrow H_0 = -\frac{\mu}{\mu_0} H_i$$

$$H_0 - H_i = K_f \Rightarrow \frac{-\mu_0}{\mu_0} H_i - H_i = K_f$$

$\vec{H} = H_0 \hat{j}$ above plane.

$\vec{B} = B_0 \hat{j}$

$\vec{H} = H_i \hat{j}$ below plane

$\vec{B} = B_i \hat{j}$ "

$$H_i = \frac{-K_f}{(1 + \frac{\mu}{\mu_0})} \qquad B_i = \frac{-\mu K_f}{(1 + \frac{\mu}{\mu_0})}$$

$$H_0 = \frac{\frac{\mu}{\mu_0} K_f}{(1 + \frac{\mu}{\mu_0})} \qquad B_0 = \frac{\mu K_f}{(1 + \frac{\mu}{\mu_0})}$$

2005 CAP Exam

9 a) $\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad E_n = \frac{\pi^2\hbar^2}{2ma^2} n^2$

At time $t=0$, equal mix of $\phi_1(x)$ and $\phi_2(x)$

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + \phi_2(x)\right) \qquad 1/\sqrt{2} = \text{Normalization}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x) e^{-iE_1 t/\hbar} + \phi_2(x) e^{-iE_2 t/\hbar}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\phi_1(x) e^{-i\omega t} + \phi_2(x) e^{-4i\omega t}\right) \qquad \omega = E_1/\hbar = \frac{\pi^2\hbar}{2ma^2}$$

b) $\langle x \rangle = \int_0^a dx\, \Psi^*(x,t)\; x\; \Psi(x,t)$

$$= \frac{1}{2}\int_0^a \left(\phi_1 e^{i\omega t} + \phi_2 e^{4i\omega t}\right) x \left(\phi_1 e^{-i\omega t} + \phi_2 e^{-4i\omega t}\right) dx$$

$$= \frac{1}{2}\left[\int_0^a |\phi_1|^2 x\,dx + \int_0^a |\phi_2|^2 x\,dx + \int_0^a \phi_1\phi_2 \underbrace{\left(e^{3i\omega t} + e^{-3i\omega t}\right)}_{2\cos(3\omega t)} x\,dx\right]$$

Need the following integrals

(i) $\int_0^a |\phi_n(x)|^2 x\, dx = \frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) x\,dx \qquad \cos 2\theta = 1 - 2\sin^2\theta, \quad 2\sin^2\theta = 1 - \cos 2\theta$

$$= \frac{1}{a}\left[\int_0^a x\,dx - \int_0^a \cos\left(\frac{2n\pi x}{a}\right) x\,dx\right]$$

$$= \frac{a}{2} - \frac{1}{a}\int_0^a \cos\left(\frac{2n\pi x}{a}\right) x\,dx$$

$$= \frac{a}{2} - \frac{1}{a}\frac{a}{2n\pi}\int_0^a d\left(\sin\left(\frac{2n\pi}{a}x\right)\right) x$$

$$= \frac{a}{2} - \frac{1}{2n\pi}\underbrace{\left[\sin\left(\frac{2n\pi}{a}x\right)x\right]_0^a}_{=0} + \frac{1}{2n\pi}\int_0^a \sin\left(\frac{2n\pi}{a}x\right)dx$$

$$= \frac{a}{2} - \frac{1}{2n\pi}\frac{a}{2n\pi}\int_0^a d\left(\cos\left(\frac{2n\pi}{a}x\right)\right)$$

$$= \frac{a}{2} - \frac{a}{(2n\pi)^2}\left(\cos\frac{2n\pi}{a}x\right)_0^a$$

$$= \frac{a}{2} - \frac{a}{(2n\pi)^2}\underbrace{\left[\cos(2n\pi) - \cos(0)\right]}_{=0}$$

$$= a/2$$

(ii) $$\int_0^a \phi_m(x)\,\phi_n(x)\,x\,dx = \frac{2}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)x\,dx$$

$$= \frac{1}{a}\int_0^a\left[\cos\left(\frac{(m-n)\pi}{a}x\right) - \cos\left(\frac{(m+n)\pi}{a}x\right)\right]x\,dx$$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{a}{((m-n)\pi)^2}\left[\cos((m-n)\pi) - \cos(0)\right] - \frac{a}{((m+n)\pi)^2}\left[\cos((m+n)\pi - \cos(0)\right]$$

$$= \frac{a}{(m-n)^2\pi^2}\left[(-)^{m-n} - 1\right] - \frac{a}{(m+n)^2\pi^2}\left[(-)^{m+n} - 1\right]$$

$$= \frac{-2a}{(m-n)^2\pi^2} + \frac{2a}{(m+n)^2\pi^2} \quad \text{if } m+n = \text{odd}$$

$$= \begin{cases} = \dfrac{2a}{\pi^2}\left(\dfrac{1}{(m+n)^2} - \dfrac{1}{(m-n)^2}\right) & \text{if } m+n = \text{odd} \\ 0 & \text{if } m+n = \text{even}\end{cases}$$

$$\langle x \rangle = \frac{1}{2}\left[\frac{a}{2} + \frac{a}{2} + 2\cos(3\omega t)\frac{2a}{\pi^2}\left(\frac{1}{9}-1\right)\right]$$

$$= \frac{a}{2} - \cos(3\omega t)\frac{2a}{\pi^2}\frac{8}{9}$$

$$= \frac{a}{2}\left(1 - \frac{32}{9\pi^2}\cos(3\omega t)\right)$$

Maximum value of $\langle x \rangle$ occurs when $\cos(3\omega t) = -1$

$$\langle x \rangle_{max} = \frac{a}{2}\left(1 + \frac{32}{9\pi^2}\right)$$

(c) Presumably here we are looking for a different mix of $\phi_1(x)$ and $\phi_2(x)$. Write

$$\Psi(x,t) = c_1\,\phi_1(x)e^{-i\omega t} + c_2\,\phi_2(x)\,e^{-4i\omega t} \qquad |c_1|^2+|c_2|^2 = 1$$

$$\langle x \rangle = |c_1|^2\int_0^a |\phi_1|^2 x\,dx + |c_2|^2\int_0^a |\phi_2|^2 x\,dx + 2c_1c_2\cos(3\omega t)\int \phi_1\phi_2\, x\,dx$$

$$= |c_1|^2\frac{a}{2} + |c_2|^2\frac{a}{2} + 2c_1c_2\cos(3\omega t)\frac{2a}{\pi^2}\left(\frac{1}{9}-1\right)$$

$$= \frac{a}{2} - c_1c_2\cos(3\omega t)\frac{32a}{9\pi^2}$$

At $t=0$

$$\langle x \rangle_{t=0} = \frac{a}{2} - c_1c_2\frac{32a}{9\pi^2}$$

Find c_1c_2 such that $\langle x \rangle_{t=0} = \frac{a}{2}\left(1+\frac{32}{9\pi^2}\right)$

$$\frac{a}{2} - c_1c_2\frac{32a}{9\pi^2} = \frac{a}{2} + \frac{1}{2}\cdot\frac{32a}{9\pi^2}$$

$$c_1c_2 = -1/2$$

$$c_1 = \frac{1}{\sqrt{2}} \qquad c_2 = -\frac{1}{\sqrt{2}}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\phi_1(x)e^{-i\omega t} - \phi_2(x)e^{-i4\omega t}\right]$$

(d) Expectation value of the momentum is obtained from Ehrenfest's theorem

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$

$$\langle p\rangle = m \cdot c_1 c_2 \, 3\omega \sin(3\omega t) \frac{32a}{9\pi^2}$$

$$= c_1 c_2 \, m\omega \frac{32a}{3\pi^2} \sin(3\omega t) \qquad m\omega = \frac{\pi^2\hbar}{2a^2}$$

$$= c_1 c_2 \frac{\pi^2\hbar}{2a^2} \cdot \frac{32a}{3\pi^2} \sin(3\omega t)$$

$$= c_1 c_2 \frac{16\hbar}{3a} \sin(3\omega t)$$

Put $c_1 c_2 = -1/2$

$$\langle p\rangle = -\frac{8\hbar}{3a}\sin(3\omega t)$$

Maximum value occurs when $\sin(3\omega t) = -1$

$$\langle p\rangle_{max} = \frac{8\hbar}{3a}$$

MJB

10. I'm guessing that it's a complex FT that they want

with $f(t) = 0 \quad t<0$
$= Ae^{-at}\cos\omega_1 t, \ t>0$

a) $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, f(t) e^{i\omega t}$

$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dt\, \frac{A}{2} e^{-at} (e^{i\omega_1 t} + e^{-i\omega_1 t}) e^{i\omega t}$

$= \frac{A}{2\sqrt{2\pi}} \left[-\frac{1}{-a+i(\omega+\omega_1)} - \frac{1}{-a+i(\omega-\omega_1)} \right]$

$= \frac{A}{2\sqrt{2\pi}} \left[\frac{1}{a-i(\omega+\omega_1)} + \frac{1}{a-i(\omega-\omega_1)} \right]$

b) Must be many oscillations during time $\sim 1/a$ ie $\omega_1 \gg a$. Neglecting term in $\omega+\omega_1$ which gives a pole at $\omega = -\omega_1$,
Power spectrum $P(\omega) \propto |F(\omega)|^2$

$$\simeq \left(\frac{A}{2\sqrt{2\pi}}\right)^2 \frac{1}{(\omega-\omega_1)^2 + a^2}$$

peaking at $\omega = +\omega_1$ with width $\simeq a$.

c) $a \ll \omega_{beat} \ll \omega_1$

d) Now $F(\omega) \simeq \frac{A}{2\sqrt{2\pi}} \left[\frac{1}{a-i(\omega-\omega_1)} + \frac{1}{a-i(\omega-\omega_2)} \right]$

$P(\omega) = |F(\omega)|^2$ get complicated — haven't got a simple form yet — but note $|\omega_1-\omega_2| \gg a$ — still complicated.