2007 CAP EXAM

QUESTION 1

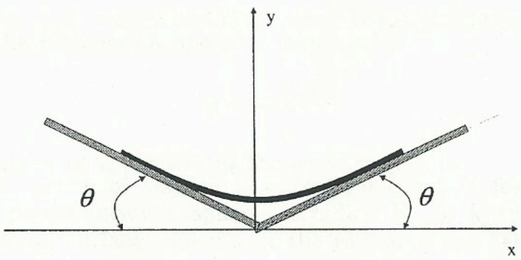
A rope of length L and mass m rests on two platforms which are both inclined at an angle θ (which you are free to pick), as shown below. The rope has a uniform mass density, and its coefficient of friction with the platforms is μ_s=1. The system is symmetrical about the y-axis.

(a) Draw a free-body diagram for the rope. (Hint: Consider the fraction of the rope that is not in contact with the platforms and the segments in contact with the platforms separately.)

(b) Given that the rope is in equilibrium, write down Newton's 1^{st} law in component form.

(c) Using your results from part (b), find an expression in terms of θ for the fraction of the rope f that does not touch the platforms.

(d) For what angle θ is the fraction f of rope not touching the surface maximized?



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QUESTION 6

The density in an ocean of depth H varies with depth as $\rho = \rho_0(1 + cz)$, where ρ_0 is the density at the surface, c is a constant, and the z-axis is directed downwards. (Typically, the density varies due to different amounts of salt dissolved in the water at different depths. The lighter fluid is on top of heavier fluid providing for static stability of the fluid column).

(a) Find the distribution of hydrostatic pressure with depth.

(b) Find the total mass of a fluid column of unit horizontal cross sectional area and the position of its centre of mass. Is the center of mass above or below the middepth $(H/2)$ of the column?

(c) If the water column is mixed (by waves, turbulence etc.), what is the density of the now uniform water? What is the change in potential energy of the water column?

(d) Suggest a density distribution that minimizes the potential energy of the column given that the total mass must be conserved and the density must vary between its minimum value ρ_0 at the surface and a maximum value at the bottom.

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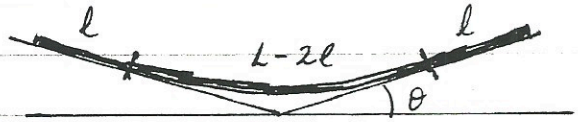
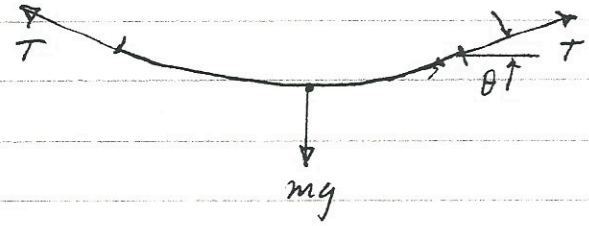
1. (a), (b)



Portions in contact have length l.

$$f = \frac{L-2l}{L} = 1 - 2\frac{l}{L}, \quad \frac{l}{L} = \frac{1}{2}(1-f)$$

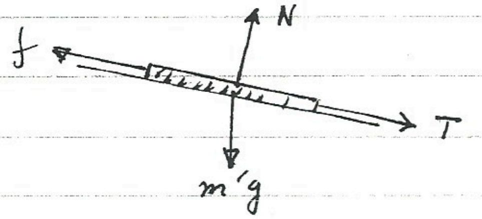
Assume the rope can only be under tension. Then the FBD for the suspended portion is



$$2T\sin\theta = mg = \lambda f L g$$

where λ = mass/unit length

For the portion in contact with the ramp,



$$N = m'g\cos\theta = \lambda l g\cos\theta$$
$$= \lambda L g \cdot \tfrac{1}{2}(1-f)\cos\theta$$

$$f = T + m'g\sin\theta = T + \tfrac{1}{2}\lambda L g(1-f)\sin\theta$$

(c) If the rope is not to slip, we require $f < f_{MAX} = \mu_s N$. Considering the limiting case, which corresponds to the maximum possible weight of the suspended portion that can be supported, we have $f = f_{MAX} = \mu_s N = N$ with $\mu_s = 1$.

$$\therefore \quad \tfrac{1}{2}\lambda L g(1-f)\cos\theta = T + \tfrac{1}{2}\lambda L g(1-f)\sin\theta$$

$$\cancel{\tfrac{1}{2}}\cancel{\lambda}\cancel{L}\cancel{g}(1-f)\cos\theta = \frac{\cancel{\lambda} f \cancel{L}\cancel{g}}{\cancel{2}\sin\theta} + \cancel{\tfrac{1}{2}}\cancel{\lambda}\cancel{L}\cancel{g}(1-f)\sin\theta$$

$$f\left(\frac{1}{\sin\theta} - \sin\theta + \cos\theta\right) = \cos\theta - \sin\theta$$

$$f = \frac{\cos\theta\sin\theta - \sin^2\theta}{\cos\theta\sin\theta - \sin^2\theta + 1}$$

$$= \frac{\sin 2\theta + \cos 2\theta - 1}{\sin 2\theta + \cos 2\theta + 1} \equiv \frac{g-1}{g+1}$$

This gives the largest value of f for a given angle θ. To maximize f,

$$f' = \frac{g'}{g+1} - \frac{(g-1)g'}{(g+1)^2} = \frac{2g'}{(g+1)^2} = 0$$

$$\text{i.e. } g' = 2\cos 2\theta - 2\sin 2\theta = 0$$

$$\tan 2\theta = 1$$

$$\therefore 2\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8} \quad (22.5°)$$

$$f_{MAX} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 0.17$$

The maximum fraction of the rope that can be suspended is ~ 17%. This result of course is for the special case $\mu_s = 1$.

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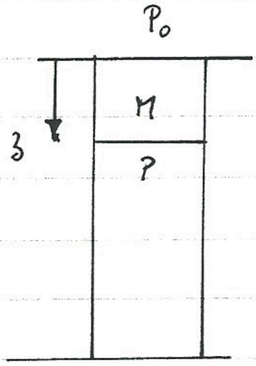
6. (a)



The mass M is supported by the pressure at depth z:

$$PA = P_0 A + Mg$$

where $$M = \int_0^z \rho_0 (1 + c z') A \, dz'$$

$$= \rho_0 A \left(z + \tfrac{1}{2} c z^2 \right)$$

$$\therefore \; P(z) = P_0 + \rho_0 g z \left(1 + \tfrac{1}{2} c z \right)$$

(b) $$M(H) = \rho_0 A H \left(1 + \tfrac{1}{2} c H \right)$$

$$\frac{M(H)}{A} = \rho_0 H \left(1 + \tfrac{1}{2} c H \right)$$

$$\bar{z} = \frac{\int_0^H z \, dm}{\int_0^H dm} = \frac{\int_0^A z \cancel{\rho_0} (1 + c z) \cancel{A} \, dz}{\int_0^H \cancel{\rho_0} (1 + c z) \cancel{A} \, dz}$$

$$= \frac{\tfrac{1}{2} H^2 + \tfrac{1}{3} c H^3}{H + \tfrac{1}{2} c H^2}$$

$$= \tfrac{1}{2} H \, \frac{1 + \tfrac{2}{3} c H}{1 + \tfrac{1}{2} c H}$$

$$> \tfrac{1}{2} H \qquad \text{since } \tfrac{2}{3} > \tfrac{1}{2}.$$

$\therefore$ the centre of mass is <u>below</u> ~~mid~~ mid-depth.

(c) $\bar{\rho} = \frac{M}{V} = \frac{\rho_0 AH(1+\frac{1}{2}cH)}{AH} = \rho_0(1+\frac{1}{2}cH)$

Take $z = H$ as the zero of potential energy, the potential energy of an element at depth z is

$$dU = dm\, g\,(H - z)$$

$$= \rho(z)\, A\, dz\, g\,(H - z)$$

For the original density distribution,

$$U = \int_0^H \rho_0 (1+cz) A g (H - z)\, dz$$

$$= \rho_0 A g \left(H^2 + \tfrac{1}{2} c H^3 - \tfrac{1}{2} H^2 - \tfrac{1}{3} c H^3 \right)$$

$$= \tfrac{1}{2} \rho_0 A g H^2 \left(1 + \tfrac{1}{3} c H \right)$$

For the mixed column,

$$\bar{U} = \int_0^H \bar{\rho} A g (H - z)\, dz$$

$$= \bar{\rho} A g \left(H^2 - \tfrac{1}{2} H^2 \right)$$

$$= \tfrac{1}{2} \bar{\rho} A g H^2$$

$$= \tfrac{1}{2} \rho_0 A g H^2 \left(1 + \tfrac{1}{2} c H \right) = \tfrac{1}{2} M(H) g H$$

$$\therefore \; U < \bar{U}$$

This is consistent with the centre of mass being below $\frac{H}{2}$.

(d) To decrease the potential energy further, it is clear that we should try to lower the centre of mass further. This can be achieved by having the density increase with depth faster than linear, i.e.,

$$\rho(z) = \rho_0 (1 + c' z^n) \quad \text{with } n > 1.$$

The total mass in this case is

$$M = \int_0^H \rho_0 (1 + c' z^n) A \, dz$$

$$= \rho_0 A \left(H + \frac{c'}{n+1} H^{n+1} \right)$$

$$= \rho_0 A H \left(1 + \frac{c'}{n+1} H \right)$$

If this is to be held fixed, we require

$$1 + \frac{c'}{n+1} H = 1 + \frac{c}{2} H$$

$$\text{i.e. } c' = \frac{n+1}{2} c$$

$$\therefore \rho(z) = \rho_0 \left(1 + \frac{n+1}{2} c z^n \right)$$

One could then calculate the potential energy which can be ~~and try to~~ minimized with respect to n.