Problem 1
Find the period of small oscillations for the planar pendulum shown in Fig. 1. The wire has a total length $\pi R < L < 2\pi R$, and is nailed to the fixed frame of radius R at point P. Assume that the portion of the wire not touching the frame remains straight, as shown and ignore friction.

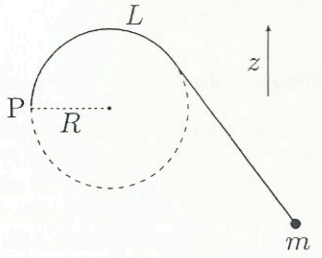


FIG. 1: planar pendelum - z is the vertical direction

Problem 2
A block of right-angle triangular cross-section and mass M is sliding on a frictionless floor while a homogeneous cylinder of mass m and radius R is rolling without slipping down a side of the block oriented at an angle α from the horizontal. (See figure.) Assume that initially both block and cylinder are at rest with the point of contact of the cylinder and the block at a height $H \gg R$ above the floor.

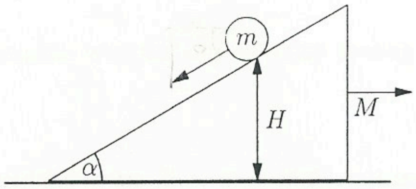


FIG. 2: block and cylinder on floor

a) Find the equations of motion.

b) After how much time will the cylinder reach the floor?

Problem 3
A particle interacts with a background scalar field which affects its inertial mass. In a particular inertial frame, the scalar field appears static and the particle's rest mass depends on its position x via $m(x) = m_0 e^{-\alpha x}$, $\alpha > 0$. At time $t = 0$, the particle is at rest at $x = 0$. At later times it is moving to the right with a non-zero velocity. Compute its subsequent trajectory, using relativistic mechanics.

Problem 4
Optical traps that are made up of interfering laser beams have recently been used to capture and confine ultra cold atoms. Near the centre of an optical trap, laser beams produce an effective electric field of the form

$$\vec{E}(x) = E_0(1 - x^2/x_0^2)\hat{e}_z \tag{1}$$

where typically, $E_0 = 5,000$ V/m, $x_0 = 5\mu$m and x is the distance from the centre of the trap. A rubidium atom (^{87}Rb) moving along the x-direction with 0.1 mm/s speed is located at $x = 0$ when the trap is turned on. According to the periodical table, this isotope has 37 protons, 50 neutrons and one 5s electron; the atomic radius is about 2.5A.

a) Treating the atom as a point nucleus surrounded by a neutralizing uniformly charged sphere, calculate its polarizability (electric dipole moment divided by electric field).

b) Describe quantitatively the motion of this rubidium atom after the trap is turned on. (Find the time and length scale of the motion of this atom in the trap; specify the assumptions used in the problem.)

c) In order to be captured by the trap, what is the maximal speed a rubidium atom can have when the trap is turned on?

Problem 5
Consider a piece of glass (with index of refraction $n = 1.5$) with normally incident light at wavelength 530 nm.

a) Design a single dielectric layer which could be applied to the glass that acts as an anti-reflection coating. In other words, what is the thickness (d) and index of refraction (n_f) of a single dielectric layer that would prevent any of the light from reflecting back to the left, towards the source? Assume the glass is infinitely thick. (To solve this problem you will need to consider the multiple reflections from the air-film interface and the film-glass interface. You may assume all media are lossless nonmagnetic dielectrics.) (See figure.)

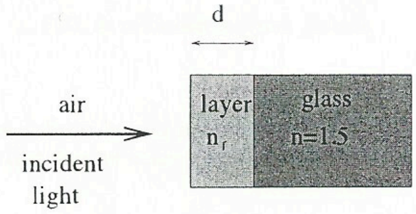


FIG. 3: anti-reflection coating

b) Based on your answer to part a) explain why the anti-reflection coating on your eyeglasses looks purple when viewed at an angle.

Problem 6
A particular quantum system can be modeled by the illustrated configuration of balls and springs, where m and k are the masses of the balls and the spring constants of the springs and the balls are constrained to move in the x-direction only.

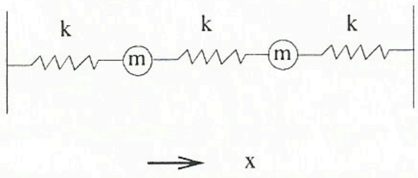


FIG. 4: quantum balls and springs. The outer ends of the springs are fixed.

a) Determine the spectrum of energy eigenvalues of this system, assuming that the classical energy in the equilibrium configuration shown is zero.

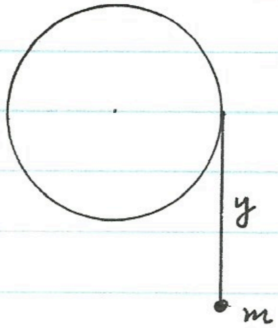
b) What is the expected value of $(l - l_0)^2$ in the ground state where l and l_0 are the distance between the balls and the equilibrium distance between the balls respectively?

Problem 7
A protein may be modeled as a particular configuration of a collapsed polymer globule (see figure). Let this particular polymer consist of N residues (or beads) connected by $N - 1$ bonds. Let there be ν orientational states per bond.

a) If the reorientation time between bond states is 10^{-12} s, estimate the time it would take for the polymer to find the particular "native" conformation of the protein by random search through conformations. For this part of the problem, let $\nu = 10$, and $N = 101$. Is your estimate reasonable? (i.e. is this a reasonable time scale for a protein to fold?)

2009 CAP Exam

Q.1.



In equilibrium the mass m hangs straight down and the free length of the string is

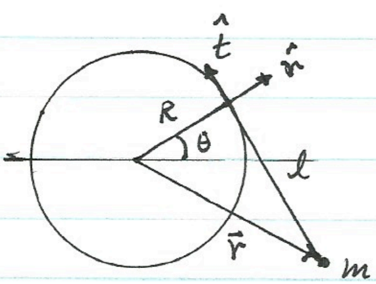
$$y = L - \pi R$$

One might guess that this is a simple pendulum with period

$$T = 2\pi\sqrt{\frac{y}{g}}$$

This is correct.

<u>Solution</u>



$$\ell = L - R(\pi - \theta) \quad = y + R\theta$$

$$\vec{r} = R\hat{r} - \ell\hat{t}$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{t} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

The velocity of the mass is

$$\vec{v} = \dot{\vec{r}} = R\dot{\hat{r}} - \ell\dot{\hat{t}} - \dot{\ell}\hat{t}$$

$$\dot{\hat{r}} = (-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\,\dot{\theta} = \dot{\theta}\,\hat{t}$$

$$\dot{\hat{t}} = (-\cos\theta\,\hat{x} - \sin\theta\,\hat{y})\,\dot{\theta} = -\dot{\theta}\,\hat{r}$$

$$v^2 = R^2\,\dot{\hat{r}}\cdot\dot{\hat{r}} + \ell^2\,\dot{\hat{t}}\cdot\dot{\hat{t}} + \dot{\ell}^2\,\hat{t}\cdot\hat{t} - 2R\ell\,\dot{\hat{r}}\cdot\dot{\hat{t}} - 2R\dot{\ell}\,\dot{\hat{r}}\cdot\hat{t}$$
$$+ 2\ell\dot{\ell}\,\dot{\hat{t}}\cdot\hat{t}$$
$$= R^2\dot{\theta}^2 + \ell^2\dot{\theta}^2 + \dot{\ell}^2 - 2R\dot{\ell}\dot{\theta}$$
$$= R^2\dot{\theta}^2 + \ell^2\dot{\theta}^2 + R^2\dot{\theta}^2 - 2R^2\dot{\theta}^2$$
$$= \ell^2\dot{\theta}^2$$

We use an energy method to solve the oscillation problem

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgh$$

where

$$\begin{aligned} h &= r_y - (-y) \\ &= R\sin\theta - l\cos\theta + y \\ &\approx R\theta - (y + R\theta)(1 - \tfrac{1}{2}\theta^2 + \cdots) + y \\ &= R\theta - y - R\theta + \tfrac{1}{2}y\theta^2 + y \\ &= \tfrac{1}{2}y\theta^2 \quad \text{for small } \theta. \end{aligned}$$

$$\begin{aligned} \therefore\ E &= K + U \\ &= \tfrac{1}{2}m(y + R\theta)^2\dot{\theta}^2 + \tfrac{1}{2}mgy\theta^2 \\ &\approx \tfrac{1}{2}my^2\dot{\theta}^2 + \tfrac{1}{2}mgy\theta^2 \end{aligned}$$

$$\dot{E} = 0 = my^2\dot{\theta}\ddot{\theta} + mgy\theta\dot{\theta}$$

$$\ddot{\theta} + \frac{g}{y}\theta = 0$$

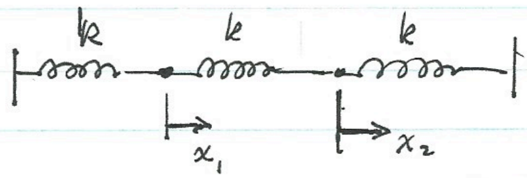
$$\therefore\ \omega = \sqrt{\frac{g}{y}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{y}{g}}$$

This is the answer guessed at the beginning!

CAP 2009

#6 (a)



$$V = \tfrac{1}{2}k x_1^2 + \tfrac{1}{2}k x_2^2 + \tfrac{1}{2}k(x_2 - x_1)^2$$
$$= k(x_1^2 + x_2^2) - k x_1 x_2$$

$$x = x_2 - x_1 \qquad X = x_2 + x_1$$
$$x^2 = x_2^2 + x_1^2 - 2x_1x_2, \quad X^2 = x_2^2 + x_1^2 + 2x_1x_2$$

$$x^2 + X^2 = 2(x_1^2 + x_2^2)$$
$$X^2 - x^2 = 4x_1x_2$$

$$\therefore V = \tfrac{1}{2}kX^2 - \tfrac{1}{4}k(X^2 - x^2)$$
$$= \tfrac{1}{4}kX^2 + \tfrac{3}{4}kx^2$$

$$H\psi(x_1, x_2) = E\psi(x_1, x_2)$$

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V(x_1, x_2)$$

Change to x, X variables

$$\frac{\partial}{\partial x_1} = \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} + \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} = -\frac{\partial}{\partial x} + \frac{\partial}{\partial X}$$

$$\frac{\partial^2}{\partial x_1^2} = \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial X}\right)\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial X}\right) = \frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial X^2}$$

Similarly, $$\frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial X^2}$$

$$\therefore -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)$$

$$= -\frac{\hbar^2}{2m}\left(2\frac{\partial^2}{\partial x^2}+2\frac{\partial^2}{\partial X^2}\right)$$

$$= -\frac{\hbar^2}{2(\frac{m}{2})}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2(\frac{m}{2})}\frac{\partial^2}{\partial X^2}$$

$$\therefore H = -\frac{\hbar^2}{2(\frac{m}{2})}\frac{\partial^2}{\partial x^2} + \frac{3}{4}kx^2 - \frac{\hbar^2}{2(\frac{m}{2})}\frac{\partial^2}{\partial X^2} + \frac{1}{4}kX^2$$

Have two independent harmonic oscillators:

$$m_1 = \frac{m}{2}, \quad k_1 = \frac{3}{2}k, \quad \omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{3}\sqrt{\frac{k}{m}} = \sqrt{3}\,\omega_0$$

$$m_2 = \frac{m}{2}, \quad k_2 = \frac{1}{2}k, \quad \omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k}{m}} = \omega_0$$

$$\therefore E = (n_1+\tfrac{1}{2})\hbar\omega_1 + (n_2+\tfrac{1}{2})\hbar\omega_2$$

$$= (n_1+\tfrac{1}{2})\hbar\sqrt{3}\,\omega_0 + (n_2+\tfrac{1}{2})\hbar\omega_0$$

Ground state energy is $E_0 = \left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\hbar\omega_0$

(b) $(l-l_0)^2 = (x_2-x_1)^2 = x^2$

So we require $\langle x^2\rangle$

$$\psi(x) = A\,e^{-x^2/\ell^2} \qquad \text{where } \ell = \sqrt{\frac{\hbar}{m_1\omega_1}}$$

$$\langle x^2\rangle = \frac{\int dx\, x^2 e^{-x^2/\ell^2}}{\int dx\, e^{-x^2/\ell^2}}$$

$$I(\alpha) = \int dx\, e^{-\alpha x^2} \qquad \alpha x^2 = y^2$$

$$x = \frac{1}{\sqrt{\alpha}} y$$

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} dy\, e^{-y^2}$$

$$= \sqrt{\frac{\pi}{\alpha}}$$

$$\frac{\partial I}{\partial \alpha} = -\int dx\, x^2 e^{-\alpha x^2}$$

$$\therefore \int dx\, x^2 e^{-\alpha x^2} = -\frac{\partial I}{\partial \alpha} = -\sqrt{\pi}\left(-\frac{1}{2}\right)\alpha^{-3/2} = \frac{\sqrt{\pi}}{2}\frac{1}{\alpha^{3/2}}$$

With $\alpha = \frac{1}{\ell^2}$,

$$\langle x^2 \rangle = \frac{\frac{\sqrt{\pi}}{2}\ell^3}{\sqrt{\pi}\,\ell} = \frac{1}{2}\ell^2 = \frac{1}{2}\frac{\hbar}{m_1 \omega_1}$$

$$= \frac{1}{\cancel{2}}\frac{\hbar}{\frac{m_2}{\cancel{2}}\sqrt{3}\,\omega_0}$$

$$= \frac{1}{\sqrt{3}}\ell_0^2 \qquad \ell_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$