Introduction
1. Calculators are allowed
2. Answer each question in a different booklet, with the question number, your name and that of your university on the first page.
3. The value attributed to each question takes into account the length and difficulty of the problem.
4. You are not expected to answer all questions. Please try as many as you can.

I(10)
We design a long range transport system by digging straight line tunnels between surface destinations. Gravitational attraction provides all the needed acceleration and deceleration. High vacuum and magnetic levitation guarantee that no friction acts on the trains in the tunnels.

Québec  
  
Paris

How long would it take to travel from Québec to Paris, from Québec to Toronto?

II(10)
Imagine a thermodynamic cycle followed in the 123 direction, where 12 is an adiabatic process, 23 an isothermic one and 31 an isochore decompression. This cycle represents the transformations of one mole of a real gas in a machine (it remains gaseous).
1. Draw this cycle in the p-V plane
2. If the van der Walls equation is a good representation of this gas, what is the efficiency of this machine
3. Which of the two real gas effects affects most this efficiency, the finite volume of the molecules or the interaction between the molecules?
III(10)

It is possible to view a dielectric medium as made up of atoms on a lattice. Each atom has one electron of charge $e$ bound to the site by a potential which we approximate by an harmonic potential of frequency $\omega$ in a Hamiltonian operator $H_0$. There are $N$ atoms per unit volume. The dielectric medium is placed in an electric field $E \hat{z}$ (in the Oz direction). The field induces a polarisation (dipole) $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ in the medium, where $\chi_e$ is the electric susceptibility of the medium.

1. Give an expression for $\chi_e$ as a function of the parameters of the problem.
2. Does the susceptibility $\chi_e$ increase or decrease when the frequency $\omega$ of the harmonic potential increase and why is it so.

IV(10)

A system contains $N$ identical particles in a one dimensional space. They undergo interactions, which global result can be represented by an harmonic potential

$$V = \frac{m \omega^2 x^2}{2}.$$

If the particles are in a quantum regime, they can occupy the states of energy

$$\varepsilon_n = (n + 1/2) \hbar \omega, \quad n = 0, 1, 2, 3...$$

If the particles are in a classical regime, their energy is given by

$$\varepsilon = \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2}$$

1. Find the (classical) partition function for each regime.
2. Write an expression for the specific heat in each regime?
3. Do these specific heats differ at low temperature, at high temperature?

V(10)

An observer at rest with respect to the fixed distant stars sees an isotropic distribution of stars. That is, in any solid angle $d\Omega$ he sees $dN = N (d\Omega / 4\pi)$ stars, where $N$ is the total number of stars he can see.

Suppose that another observer is moving at a relativistic velocity $\beta$ in the $\hat{x}$ direction (rest frame system $S'$). What is the distribution of stars seen by this observer?

Specifically, what is the distribution function $P(\theta', \phi')$ such that the number of stars seen by this observer in his solid angle $d\Omega'$ is $P(\theta', \phi') d\Omega'$? Check to see that

$$\int_{\text{sphere}} P(\theta', \phi') d\Omega' = N$$

and that $P(\theta', \phi') \rightarrow \frac{N}{4\pi}$ as $\beta \rightarrow 0$.

Where will the observer see the stars bunch up?
VI(10)
You are asked to interpret the results obtained as a thin layer of material of index $n_2$ ($n_2>1$) is deposited on a very thick substrate (semi infinite in fact) of index $n_1$ ($n_1>1$). The surrounding medium is a vacuum ($n_1=1$). While depositing our layer (medium 2), the modulus of the reflection coefficient, $|\Gamma|$, (ratio of the amplitude of reflected wave over the amplitude of the incoming wave) was monitored as a function of the thickness of the layer. The result is reported on the figure where the vertical axis should read "Modulus of reflection coefficient $|\Gamma|$" and the horizontal should read "Normalised optical path $n_2 d_2 / \lambda_2$".

![Graph showing the relationship between $|\Gamma|$ and $n_2 d_2 / \lambda_2$](image)

a. Using the graph, determine first the index $n_1$ of the substrate and then the index $n_2$ of the material in the thin layer.

b. If the thickness of medium 2 equals $d_2 = \lambda_2 / 8$, what is the ratio of the (outside) stationary wave. Use the graphics to avoid a long calculation.

c. What would be the sign of $\Gamma$ for $d_2 = \lambda_2 / 2$? Why?

VII(8)
The following figure shows the observed lines, with $\lambda$ in angströms, in the spectrum of a certain atom of intermediary $Z$. These lines correspond to all the possible (optically allowed) transitions between the levels of two multiplets.
Determine the quantum numbers $LSJ$ characterizing these multiplets and their levels.
Explain your results.
VIII(7)

This problem finds some applications in biophysics where it can be used to model some neurophysiological processes. A spherical current source is embedded in a medium where the conductivity behaves as $1/r^2$ with respect to the center of the source. Determine the electrical potential and the charge density everywhere outside the source.

IX(10)

1. Write down the law of radioactive decay. Define the half-life and mean life of a radioactive nucleus and obtain the relation between them.

2. The nucleus $^{87}\text{Rb} (Z=37)$ decays into the ground state of $^{87}\text{Sr} (Z=38)$, with a half-life of $4.7 \times 10^{10}$ years and a maximum energy for the $\beta$ of 272 keV. Discuss briefly the difficulties you might encounter in attempting to measure this half-life.

3. Five samples of chondritic meteorites are found to have the following proportions of $^{87}\text{Rb}$, $^{87}\text{Sr}$, and $^{86}\text{Sr}$.

<table>
<thead>
<tr>
<th>Meteorites</th>
<th>$^{87}\text{Rb}/^{86}\text{Sr}$</th>
<th>$^{87}\text{Sr}/^{86}\text{Sr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modoc</td>
<td>0.86</td>
<td>0.757</td>
</tr>
<tr>
<td>Homestead</td>
<td>0.8</td>
<td>0.751</td>
</tr>
<tr>
<td>Bruderheim</td>
<td>0.72</td>
<td>0.747</td>
</tr>
<tr>
<td>Kyushu</td>
<td>0.6</td>
<td>0.739</td>
</tr>
<tr>
<td>Bath Furnace</td>
<td>0.09</td>
<td>0.706</td>
</tr>
</tbody>
</table>
Given that the nucleus \(^{86}\text{Sr}\) is not a daughter product of any long-lived radioactive nucleus, show that these data are consistent with a common primordial ratio \(^{87}\text{Sr}/^{86}\text{Sr}\) and a common age for all these meteorites and find that age.

X(10)

The Orion nebula is 1600 light years away from the Sun and is completely ionised. Its average temperature is 8500 K and its electronic density is 2000 cm\(^3\). Its diameter is estimated to be 1.6 light year (one light year is 365 light days).

a) Suppose that nebula is made of non collisional plasma (a good approximation), what is the cutoff frequency \(f_p\) (in Hz) for electromagnetic waves propagating in this nebula.

b) Two light pulses, \(P_1\) and \(P_2\), with carrier frequencies \(f_1 = 15\text{ GHz}\) and \(f_2 = 20\text{ GHz}\) go through the nebula. If \(P_1\) and \(P_2\) come into the nebula at the same time, what will be the delay (in seconds) between the two signals when they come out of the nebula. (Be more intelligent than your calculator; the binomial approximation could be useful...)

c) Refer to b). At the carrier frequency \(f_1\), what is \(\Delta\varepsilon = \varepsilon_p - 1\), the difference between the relative permittivity of the plasma and that of the vacuum.
The potential energy for a particle of mass, $m$, a distance $r$ from the centre of the earth is

$$V = \frac{1}{2} m \frac{g}{R} r^2$$

where $g =$ gravitational field at surface of earth

$R =$ radius of earth.

Consider the chord $AB$ through the earth.
Let $r_0$ be the closest distance of the chord to the centre of the earth.
Let $s$ be the distance along the chord, measured from the midpoint of the chord.

Then

$$r^2 = s^2 + r_0^2$$

The kinetic energy of a particle of mass $m$ travelling along $AB$ is

$$T = \frac{1}{2} m s^2$$

Lagrangian:

$$L = T - V$$

$$= \frac{1}{2} m s^2 - \frac{1}{2} m \frac{g}{R} r^2$$

$$= \frac{1}{2} m s^2 - \frac{1}{2} m \frac{g}{R} (s^2 + r_0^2)$$

Euler's equation of motion

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}} \right) = \frac{\partial L}{\partial s} = 0$$

$$m \dot{\ddot{s}} + m \frac{g}{R} S = 0$$

$$\ddot{s} = -\frac{g}{R} s$$

The equation of simple harmonic motion.
The particle undergoes simple harmonic motion with frequency

$$\omega = \sqrt{\frac{g}{R}}$$

The time $T_0$ for the particle to slide from one end of the tunnel to the other is half a period

$$T_0 = \frac{\pi}{\omega} = \pi \sqrt{\frac{R}{g}} = 42.2 \text{ min}$$

Notice this result is independent of the precise location of the chord. Each and every chord yields the same period of oscillation.

In the time it takes to go from Quebec to Toronto is the same as from Quebec to Paris, namely 42.2 min.
Consider the chord defined by the dimension H.

For the displacement x shown, with \( OA = r \)

\[
F = -\frac{GMm}{r^2}(\frac{r}{R})\hat{x}
\]

where \( r \) is the earth radius
\( M \) is the earth mass
\( m \) is the train mass.

But, for \( r = R \), \( F = -mg \hat{x} \)

\[
\frac{GM}{R^2} = g
\]

\[
F = -mg\left(\frac{r}{R}\right)\hat{x}
\]

\( x \)-component is \( F = -mg\left(\frac{r}{R}\right)\cos x\)

\[
= -mg\left(\frac{r}{R}\right)\left(\frac{x}{R}\right)
\]

Equation of motion: \[
m\ddot{x} = -m\left(\frac{g}{R}\right)x
\]

S.H.O. with \( T = 2\pi \sqrt{\frac{r}{g}} \)

Travel time \( \frac{T}{2} = \pi \sqrt{\frac{r}{g}} = \pi \sqrt{\frac{6.38 \times 10^6}{9.8}} \) \( \text{Time} = 42.2 \text{ mins} \) For all chords.
Choose $V_1$ and $T_1$; then $T_2$.

Find $V_2$ (1 and 2 are on an adiabat.)

\[ T \, ds = C_v \, dT + T \left( \frac{\partial p}{\partial T} \right)_V \, dV \]

\[(p + \frac{\alpha}{V^2})(V - \delta) = RT \text{ for the VDW gas} \]

\[ \left( \frac{\partial p}{\partial T} \right)_V = \frac{R}{V - \delta} \]

\[ \therefore T \, ds = C_v \, dT + \frac{RT}{V - \delta} \, dV \]

and \[ ds = \frac{C_v}{T} \, dT + \frac{R}{V - \delta} \, dV \]

\[ \therefore \text{Along the adiabat:} \]
\[
\int_1^2 \frac{C_V}{T} \, dT + \int_1^2 \frac{R}{V-V_0} \, dV = 0.
\]

\[
\therefore \int_1^2 \frac{C_V}{T} \, dT + R \ln \left( \frac{V_2}{V_1}\right) = 0.
\]

Hence \( V_2 \)

The Heat Engine.

\[
Q_H = \int_2^3 T \, ds = \int_2^3 C_V \, dT + RT_2 \int_2^3 \frac{dV}{V - V_0}
\]

\[
= 0 + RT_2 \left[ \ln \left( \frac{V_1 - V_0}{V_2 - V_0} \right) \right]
\]

\[
= T_2 \int_1^2 \frac{C_V}{T} \, dT
\]

\[
Q_L = -\int_3^1 T \, ds = -\int_3^1 C_V \, dT + 0
\]

isochore.

\[
Q_L = \int_1^2 C_V \, dT.
\]
For the van der Waals, $C_v$ is a function only of temperature. However, it can be a function of $a$ and $b$. 
# III. 1. It's unclear what is expected. Anyway...

\[ x = \frac{eF}{m \omega^2} \]

- The charges separate a distance \( x \) in an electric field \( E \): \( eE = kx \), harmonic force constant \( k = \frac{1}{m \omega^2} \).

- Dipole moment: \( \mu = e x = \frac{e^2}{m \omega^2} \), \( \vec{E} = \vec{r}E \).

\[ \alpha = \frac{e^2}{m \omega^2} \text{ is atomic polarizability.} \]

- In terms of the atomic dipole, the polarization is:

\[ P = n \mu = n \alpha E \]

\( \vec{E} \) is the local field. This is where the ambiguity arises. Assuming the local field is \( \vec{E} \), then

\[ E_0 = E + \frac{P}{3d_0} = (1 + \frac{1}{3} \alpha) E \]

The local field \( E_0 \) is not \( E \), the macroscopic field.

- The usual argument gives:

\[ P = n \mu E_0 = n \alpha (1 + \frac{1}{3} \alpha) E \]

\[ \therefore \quad E_0 = \frac{n \alpha E}{1 + \frac{1}{3} \alpha} \]
\[ X_e = \frac{N_e}{e_0^2} + \frac{N_d}{3e_0} X_e \]

\[ X_e = \frac{N_e/e_0}{1 - N_e/3e_0} \]

For \( n_e/e_0 \ll 1 \), get back \( X_e = \frac{N_d}{e_0^2} = \frac{N_d^2}{m_e^2} \).

From this expression, \( X_e \propto n_e \). This is simply due to the fact that the polarizability decreases as the electron becomes more tightly bound.

Defining \( \xi = \frac{N_d^2}{m_e^2} \), the full expression gives:

\[ X_e = \frac{\xi}{1 - \frac{1}{3} \frac{\xi}{\omega}} \]

\[ = \frac{\xi}{\omega - \frac{1}{3} \xi} \]

For \( \xi > \frac{\omega}{\sqrt{3}} \), \( X_e \) is positive and diverges at \( \omega = \frac{\omega}{\sqrt{3}} \). Thus, as \( \omega \) is reduced from above \( \xi \), \( X_e \) increases. The divergence is a signature of a polarizability catastrophe (phase transition).

I find it hard to believe this is what students were supposed to do!

- [Signature]
Now:

For \( S \)

\[
dN = \frac{dn}{4\pi} \frac{n}{\cos \theta} \text{ in solid angle } d\Omega
\]

Now for \( S' \)

\[
d\Omega' = d\Omega \gamma^2 \left(1 + \beta \cos \theta' \right)^2
\]

Since:

\[
d\Omega' = \sin \theta' \, d\theta' \, d\phi'
\]

and \( d\phi' = d\phi \) in the perpendicular plane

\[\cos \theta' = \cos \theta - \beta^2 \frac{1}{1 - \beta \cos \theta} \]

by aberration formula

\[
\sin \theta' \, d\theta' = \sin \theta \, d\theta \gamma^2 \left(1 + \beta \cos \theta' \right)^2
\]

Hence:

Since \( N \) is the same for \( S' \)
\[ dN = \frac{N}{4\pi} \frac{d\Omega'}{\gamma^2 \left(1 + \beta \cos \theta'\right)^2} \]

\[ \text{c.o.} \quad \frac{dN}{d\Omega'} = \frac{dN}{d\Omega} \frac{1}{\gamma^2 \left(1 + \beta \cos \theta'\right)^2} \]

(a) \quad \text{Large as } \theta' \to \frac{\pi}{2}, \quad \beta \gg 1

(b) \quad \int \frac{dN}{d\Omega'} d\Omega' = \int \frac{dN}{d\Omega} \int \frac{1}{\gamma^2 \left(1 + \beta \cos \theta'\right)^2} \sin \theta' d\theta' d\phi

= \frac{N}{4\pi \gamma^2} \cdot 2\pi \cdot \left[ \frac{1}{\left(1 + \beta x\right)^2} \right]_1^-\infty

= \frac{N}{2\gamma^2} \cdot \frac{1}{\beta} \left[ \frac{1}{1 - \beta} - \frac{1}{1 + \beta} \right]

= \frac{N}{2\gamma^2 \beta} \cdot 2\beta \gamma^2 = \frac{N}{\gamma^2 \beta}
Intermediate $J$ suggests LS coupling to account for the spin orbit interaction.

The spin-orbit energy which splits a multiplet is then:

$$< A \ L \ S > = < A \ (\frac{J^2}{2} - L^2 - \frac{S^2}{2}) >$$

$$= A\ h^2 (J(J+1) - L(L+1) - S(S+1))$$

For a given $L$ and $S$, $J = L+S, L+S-1, \ldots, L-S$.

The selection rules for optical ( allowed ) transitions between states of different multiplets are

$\Delta S = 0, \Delta J = \pm 1, 0$ (and $\Delta L = \pm 1, 0$)

The first step with the data is to calculate the energies of the spectrum lines and to look for a pattern in the energy differences. You note

$$\frac{\Delta E_{12}}{\Delta E_{13}} \approx \frac{2}{3}, \quad \frac{\Delta E_{14}}{\Delta E_{40}} \approx \frac{1}{2}$$
\( S = 0 \) or \( \frac{1}{2} \) will not give enough lines. For \( S = 1 \) we have

\[
L=2 \quad \text{\{3\} } \quad 3 \quad 2.80757 \\
S=1 \quad \text{\{2\} } \quad 2 \quad 2.80588 \\
\quad \quad 1 \quad 2.80642 \\
\text{L=1 } \quad \text{\{2\} } \quad \text{2 } \quad 0.01964 \\
S=1 \quad \text{\{2\} } \quad 1 \quad 0.00648 \\
\quad \quad 0 \quad 0
\]

This choice \( L=2, L=1 \) fits the given pattern except for the energy difference \( \Delta E_{45} \) which should be equal to \( \Delta E_{12} \) but is somewhat off.

MFS
a spherical current source, $I$, requires

$$\mathbf{J}(r) = \frac{I}{4\pi r^2} \hat{r}, \text{in steady-state}$$

where $\mathbf{J}$ is the current density

$$\mathbf{J} = \sigma \mathbf{E}, \text{ Ohm's Law}$$

where $\sigma$ is the conductivity and $\mathbf{E}$ the electric field, $\sigma$ behaves as $\frac{1}{r^2}$ in the problem

$$\mathbf{J}(r) = \frac{I}{4\pi r^2} = \frac{\sigma_0 E(r)}{r^2}$$

we see that $E = \frac{I}{4\pi \sigma_0} \hat{r}$ is constant everywhere (outside the source).

$$V = -\int_{R}^{r} \mathbf{E} \cdot d\mathbf{r} = -\frac{I}{4\pi \sigma_0} (r - R)$$

$$\iint \mathbf{E} \cdot d\mathbf{A} = \iiint_{V} \rho(r) \, dV$$

$$\frac{I}{4\pi \sigma_0} = \frac{4\pi}{\varepsilon_0} \int \rho(r) r^2 \, dr$$

$$\frac{2r}{\sigma_0} = \frac{4\pi}{\varepsilon_0} \rho(r) r^2$$

$$\rho(r) = \frac{\varepsilon_0 I}{2\pi \sigma_0 r}$$

From March Chen.
\[
\begin{align*}
E_1 & = E_0 + \Delta E_0 \\
E_2 & = E_0 + \Delta E_2 \\
E_3 & = E_0 + \Delta E_3 \\
E_4 & = E_0 + \Delta E_4 \\
E_5 & = E_0 + \Delta E_5 \\
E_6 & = E_0 + \Delta E_6 \\
E_7 & = E_0 + \Delta E_7 \\
E_8 & = E_0 + \Delta E_8 \\
E_9 & = E_0 + \Delta E_9 \\
\end{align*}
\]
Then \( d = 0.25 \frac{a}{\lambda_2} \)

\[
\begin{align*}
E_1 + E_2 &= E_3 + E_4 \\
E_1 - E_2 &= n_2 (E_3 - E_4) \\
-\frac{i}{\lambda_2} (E_3 - E_4) &= E_5 \\
-\frac{i}{\lambda_2} (E_3 + E_4) &= n_3 \frac{\lambda_2}{\lambda_2} E_5 \\
\therefore E_1 + E_2 &= -\frac{2n_3}{\lambda_2} E_5 \\
E_1 - E_2 &= -\frac{i}{\lambda_2} E_5 \\
\therefore E_1 &= -\frac{2n_3}{\lambda_2} E_5 \\
E_2 &= -\frac{i}{\lambda_2} (n_2 - n_3) E_5
\end{align*}
\]

\[
\frac{E_2}{E_1} = \frac{n_2 - n_3}{n_2 + n_3} = 0 \text{ for good}
\]

\[
\therefore n_2 = \frac{n_3}{n_2} \\
n_2^2 = n_3 \\
\therefore n_2 = \sqrt{3}
\]

Then \( d = 0.5 \frac{a}{\lambda_2} \)

\[
\begin{align*}
E_1 + E_2 &= E_3 + E_4 \\
E_1 - E_2 &= \sqrt{3} (E_3 - E_4) \\
E_3 + E_4 &= -E_5 \\
E_3 - E_4 &= -\sqrt{3} E_5 \\
\therefore E_1 + E_2 &= -E_5 \\
E_1 - E_2 &= -3E_5 & \therefore E_1 &= -4E_5 \\
E_2 &= 2E_5
\end{align*}
\]

\[
\frac{E_2}{E_1} = 0.5 \quad \therefore \text{ negative sign}
\]
b) From graph, \( \eta \) for \( \frac{d\eta}{\lambda_2} = \frac{1}{8} \) is about 0.35

\[ \frac{E_2}{E_1} = -0.35 \]

\[ |E_1 + |E_2| | \approx 1.35 \]

\[ |E_1 - |E_2| | \approx 0.65 \]

- Standing wave ratio = 2
Radioactive decay law: rate of change of the population is proportional to the population
\[ \frac{dN}{dt} = -\lambda N \]

\[ N(t) = N_0 e^{-\lambda t} \]

where \( N(t) = \text{population at time } t \),
\( N_0 = \text{population at time } t=0 \),
\( \lambda = \text{decay constant} \).

The half-life, \( t_{1/2} \), is the time it takes for the population to reduce by a factor of 2, viz
\[ \frac{1}{2} = e^{-\lambda t_{1/2}} \Rightarrow t_{1/2} = \frac{\ln 2}{\lambda} \]

The mean life, \( \tau \), is the average time that a nucleus is likely to survive before it decays:
\[ \tau = \frac{\int_0^{\infty} t |\frac{dN}{dt}| \, dt}{\int_0^{\infty} |\frac{dN}{dt}| \, dt} \Rightarrow \tau = \frac{1}{\lambda} \]

The activity (counts per sec.), \( A = \ln N \). Thus
\[ A(t) = A_0 e^{-\lambda t} \quad \text{where } A_0 = \ln N_0 \]

\[ \ln [A(t)] = \ln [A_0] - \lambda t \]

The usual method of measuring the half-life is to measure the activity, \( A(t) \), as a function of time over a period that would span several half-lives. A plot of \( \ln [A(t)] \) versus \( t \) should yield a straight line of slope \( -\lambda \) from which the \( t_{1/2} \) is determined. The chemical composition of the source does not have to be measured.

This method clearly fails for sources of very long half-lives, so there is no
appreciable change in the activity over the lifetime of the experiment. Instead, the activity itself is measured, $A = wN$, and then the number of atoms, $N$, has to be determined (for example, by weighing the sample whose chemical composition is accurately known).

The activity is measured by counting the electrons in the decay

$$^{87}\text{Rb} \rightarrow ^{87}\text{Sr} + e^- + \bar{\nu}_e$$

However, the electrons have very low energy, $E_e \leq 0.372$ MeV, thus there is a real possibility that the electron in collisions with other atoms will get trapped in the source and not emerge to be counted. To minimize this effect, very thin sources are used. This reduces the activity and hence the count rate-- making the experiment more difficult. Detectors capable of measuring low energy electrons are often difficult to calibrate-- another problem for the experiment.

3. Let $t_0 =$ the time at which the meteorites are formed

$t_1 =$ today's time

$t_i - t_0 =$ age of the meteorites

$N_{i1}(t) =$ number of atoms of $^{87}\text{Rb}$ at time $t$

$N_{i2}(t) =$ number of atoms of $^{87}\text{Sr}$ at time $t$

$N_3 =$ number of atoms of $^{86}\text{Sr}$

Since $^{86}\text{Sr}$ is not a daughter product of any long-lived radioactive nucleus, and is stable, the population $N_3$ is constant with time.

The population $N_1$ varies with time

$$N_{i1}(t_1) = N_{i1}(t_0) e^{-w(t_1 - t_0)}$$

However, the loss in the number of atoms of $^{87}\text{Rb}$ is matched by the gain in the number of atoms of $^{87}\text{Sr}$. That is

$$N_{i1}(t_1) + N_{i2}(t_1) = N_{i1}(t_0) + N_{i2}(t_0)$$
Divide all through by $N_3$

$$\frac{N_2(t_1)}{N_3} = \frac{N_1(t_0)}{N_3} \frac{N_1(t_1)}{N_3} + \frac{N_2(t_0)}{N_3}$$

$$ = \frac{N_1(t_1)}{N_3} e^{\frac{N_1(t_1)}{N_3}} - \frac{N_1(t_1)}{N_3} + \frac{N_2(t_0)}{N_3}$$

$$\frac{N_2(t)}{N_3} = \frac{N_1(t)}{N_3} \left[ e^{\frac{w(t-t_0)}{N_3}} - 1 \right] + \frac{N_2(t_0)}{N_3} \quad (i)$$

Assuming that the primordial ratio $^{87}Sr/^{86}Sr = \frac{N_2(t_0)}{N_3}$ is fixed, then eq. $(i)$ takes the form of

$$y = m \times + c$$

$$y = \frac{^{87}Sr}{^{86}Sr} = \frac{N_2(t_1)}{N_3} \text{ at today's time}$$

$$x = \frac{^{87}Rb}{^{86}Sr} = \frac{N_1(t_1)}{N_3} \text{ at today's time}$$

$$m = \left[ e^{\frac{w(t-t_0)}{N_3}} - 1 \right]$$

$$c = \frac{N_2(t_0)}{N_3} = \frac{^{87}Sr}{^{86}Sr} \text{ at } t = t_0$$

Plotting the given data

![Graph showing the plotted data points for $^{87}Sr/^{86}Sr$ ratio against time. The line fits the data well.]
The slope

\[ m = \frac{0.757 - 0.706}{0.86 - 0.09} = \frac{0.051}{0.77} = 0.0662 \]

\[ e^{w(t_i - t_0)} - 1 = 0.0662 \]

\[ e^{w(t_i - t_0)} = 1.0662 \]

\[ t_i - t_0 = \frac{1}{w} \ln (1.0662) \]

\[ = \frac{t_0}{\ln 2} \ln (1.0662) \]

\[ t_0 = 4.7 \times 10^{10} \text{ yr} \]

**Common age of samples** = \( 4.35 \times 10^9 \text{ yrs} \)
This problem deals with the propagation of electromagnetic waves in a plasma, that is, a region of free electrons. The cut-off frequency referred to in the problem is the plasma frequency, \( \omega_p = \frac{N e^2}{\epsilon_0 m_e} \),

which we can derive as follows.

Propagation of \( E \), \( M \) waves are governed by Maxwell's equations:

\[
\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0
\]

\[
\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{B} - \mu_0 \frac{\partial \mathbf{D}}{\partial t} = 0
\]

or \( \nabla \times \mathbf{B} - \frac{1}{\epsilon_0 c^2} \frac{\partial \mathbf{D}}{\partial t} = 0 \).
We need a relation between $D$ and $E$

\[ D = P + E \]

\[ P = \text{dipole moment/volume} \]

or single electron $m \ddot{x} = -eE$

\[ \text{+ dipole moment/electron } p = -ex \]

for harmonic wave $\tilde{E} = \tilde{E} \cos(-i\omega t)$ we have

\[ -mw^2 \ddot{x} = -eE \Rightarrow p = -e \frac{\ddot{E}}{mw^2} \]

Thus we have

\[ D = E_0 \left(1 - \frac{Ne^2}{mw^2 \varepsilon_0} \right) \tilde{E} \]

\[ = E_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) \tilde{E} \]

where $\omega_p^2 = \frac{Ne^2}{m\varepsilon_0}$

\[ E(\omega) = E_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) \]
Combining \[ \nabla \times E + \frac{\partial B}{\partial t} = 0 \]

and \[ \nabla \times B - \frac{\varepsilon(\omega)}{\varepsilon_0 c^2} \frac{\partial^2 E}{\partial t^2} = 0 \]

will of 2nd equation gives

\[- \nabla^2 B + \frac{\varepsilon(\omega)}{\varepsilon_0 c^2} \frac{\partial^2 B}{\partial t^2} = 0 \]

\[ \frac{d}{d\omega} B \sim e \quad \text{we get dispersion relation} \]

\[ k^2 - \omega^2 + \frac{\omega_p^2}{c^2} = 0 \]

\[ \omega = \left( c_k^2 + \omega_p^2 \right)^{1/2} \]

\[ \frac{U_{\text{phase}}}{k} = \frac{\omega}{k} = c \left( 1 + \frac{\omega_p^2}{c^2 k^2} \right)^{1/2} \]

\[ \frac{d\omega}{dk} = c \left( 1 + \frac{\omega_p^2}{c^2 k^2} \right)^{-1/2} < c \]
a) Calculate $f_p = 2\pi \omega_p$

$$= 2\pi \left( \frac{Ne^2}{\varepsilon_0 m_e} \right)^{\frac{1}{2}}$$

$N = 2 \times 10^{9} \text{ m}^{-3}$

$e = 1.6 \times 10^{-19} \text{ C}$

$\varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 / \text{Nm}^2$

$m_e = 9.109 \times 10^{-31} \text{ kg}$

$$f_p = 2\pi \left( \frac{2 \times 10^{9} \text{ m}^3 \cdot (1.6 \times 10^{-19})^2 \cdot \varepsilon_0}{8.85 \times 10^{-12} \cdot 9.109 \times 10^{-31}} \right)^{\frac{1}{2}}$$

$$= 1.58 \times 10^7 \text{ Hz} = 15.8 \text{ MHz}$$
1) \[ \Delta t = t_1 - t_2 \]

\[ = \frac{d}{v_1} - \frac{d}{v_2} \]

\[ = \frac{d}{c} \left( \left( 1 + \frac{1}{f_1^2} \right)^{1/2} - \left( 1 + \frac{1}{f_2^2} \right)^{1/2} \right) \]

Using Taylor expansion:

\[ = \frac{d}{c} \times \frac{1}{2} \times f_p^2 \left( \frac{1}{f_1^2} - \frac{1}{f_2^2} \right) \]

\[ = 0.8 \text{ years} \times \left( \left( \frac{15.8}{15 \times 10^3} \right)^2 - \left( \frac{15.8}{20 \times 10^3} \right)^2 \right) \]

\[ = 12.35 \text{ s} \]

c) \[ \Delta \varepsilon = \varepsilon(\omega) - \varepsilon_0 \]

\[ = -\frac{f_p^2}{\varepsilon_0 f_1^2} \]

\[ = -\left( \frac{1.58 \times 10}{1.5 \times 10^4} \right)^2 = -1.11 \times 10^{-6} \]