# Quantifying Euler-Bernoulli and Timoshenko Beam Theory Accuracy for Estimating Flexural Rigidity of a Bone Surrogate in Four-Point Bending

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INTRODUCTION: Aging and osteoporosis reduce the stiffness and strength of bone. The lifetime risk of bone fragility fracture at the hip, spine, and forearm, for a 50-year-old white woman is estimated to be 30%-40% [1]. Mechanical bend testing and modelling are performed to investigate factors that affect the flexural rigidity and strength of bones. Van Lenthe et al., 2008 [2] and Julián et al., 2020 [3] found that Euler-Bernoulli (EB) beam theory underestimates the Young's modulus of mice and rat femora, respectively. EB beam theory assumes prismatic long slender beams, which is not the case for most bones [2]. Although Timoshenko beam theory (TI) is used to assess bending properties of beams that are not slender, it has not been used to assess the flexural rigidity (EI) of bone. Timoshenko beam theory accounts for shear deformation and rotational bending effects in short thick beams. The purpose of this study was to compare TI and EB beam theories for assessing bone EI and to quantify the difference in these analytical methods for evaluation of bone stiffness. The bi-material bone surrogate, developed in a previous study [4] was used.

METHODS: Four-point bend tests on three bone surrogates were performed with six repeats. TI and EB beam theories were used to calculate the bone surrogate EI from the experimentally determined stiffness. Deformation was measured at the cross head, midpoint and with digital image correlation (DIC).
Each bone surrogate was tested six times under four-point bending to determine bending stiffness. Tests were performed using a custom-designed four-point bending apparatus and MTS Sintech 10/GL testing machine. The bone surrogates were centered in the apparatus with a distance of L=50.8 mm between supporting rollers. Displacement was applied by upper load applicators (25.4 mm apart) at a speed of 2.3 mm/min. Displacement of the bone surrogate was measured at the crosshead ($\delta_{CH}$) and with an extensometer ($\delta_{Mid}$) (Epsilon model 3540-001M-ST) at the mid-point of the bottom surface. The experimental bending stiffness ($K_{CH}$, and $K_{Mid}$) was calculated as the slope of the linear portion of the force-(F) deflection (δ) data with linear regression. Local deformation at the cylindrical supports was estimated with Hertz contact theory, to adjust and reduce errors in bending stiffness.
Deformation of the bone surrogate was measured with DIC using one camera (FLIR, Blackfly S BFS-U3-63S4, Firmware 1801.0.9.0), Spinnaker SDK (SpinView GUI) and lens (Edmund Optics, Double Gauss Focus 25 mm).
The bone surrogate's EI was calculated based on EB and TI beam theories. (Eq. 1, 2).

$$EI_{EB} = \frac{Ka(4a^2-3l^2)}{48} \quad \textbf{(Eq 1)}$$

$$EI_{TI} = \frac{Ka(4a^2-3l^2)}{48\left(1-\frac{Ka}{kGA}\right)} \quad \textbf{(Eq 2)}$$

Where K is stiffness, a is the distance between upper and lower supports (12.7 mm); L is the distance between two lower supports (50.8 mm)
Methods used to calculate the bone surrogate stiffness and EI were compared using Bland-Altman analysis. One-way ANOVA was performed to test differences in EI predictions (α=0.05) with a Tukey post-hoc test. All statistical analysis was done using OriginPro 2020.

RESULTS: By comparing DIC data with experimental and analytical results, systematic error between methods used to measure stiffness was determined (Table 1). EB beam theory over-predicted the bending stiffness by 44% while TI beam theory under-predicted the stiffness by 32%. TI beam theory predicted flexural rigidity of the bone surrogate most accurately (-21 to +13%); whereas, the EB beam theory underestimated the bone surrogate flexural rigidity in all methods (-47 to -29%) (Figure 1 and Table 2). Overall, using Hertz-adjusted crosshead displacement for EI estimation, gave a mean bias of 39% between EB and TI beam theories at EI <=1.08E+8 Nmm$^2$. The bias was higher using Hertz-adjusted midpoint deflection (52%) and increased with higher EI (Figure 1).

DISCUSSION: In this study, crosshead displacement, beam maximum deflection and DIC deformation based measures of EI were quantified for a bone surrogate. This study confirmed that EB underestimated the bone surrogate EI in all methods and there was significant statistical difference between experimental EI calculated by EB and TI beam theories. TI beam theory predicted the bone surrogate EI more accurately than EB beam theory. In conclusion, EB beam theory is a common method for experimental assessment, but we found that TI beam theory predicted the bone EI more accurately. For bone surrogate stiffness measurements, it can be concluded that EB and TI beam theories result in over- and under-predictions, respectively. Limitations of this study were: small sample sizes; other sources of compliance were not taken into account; and testing was performed on a bone surrogate with simplified geometry and materials.

SIGNIFICANCE/CLINICAL RELEVANCE: Accurate estimation of bone stiffness and flexural rigidity is crucial for investigations of bone disease, diagnosis, and treatments. The results of this study increase the accuracy of such investigations furthering efforts towards improving bone health.

References
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Table 1. Bone Surrogate Bending Stiffness and Percent Errors for each Method
The error is defined with respect to the stiffness obtained from DIC (4.92E+04 N/mm).

| Method | DIC | CH | Mid | TI | EB |
|---|---|---|---|---|---|
| Stiffness (N/mm) | 4.92E+04 | 3.83E+04 | 4.84E+04 | 3.36E+04 | 7.08E+04 |
| Error (%) | | 21.9 | 1.7 | -31.6 | 43.8 |

(CH: crosshead displacement, Mid: midpoint deflection, TI: Timoshenko beam theory, EB: Euler-Bernoulli beam theory

Table 2. Bone Surrogate Flexural Rigidity and Percent Error for each Method
The error is defined with respect to the theoretical flexural rigidity (1.34E + 08 Nmm$^2$).

| Method | TI_CH | TI_Mid | EB_CH | EB_Mid | TI_DIC | EB_DIC |
|---|---|---|---|---|---|---|
| EI (Nmm$^2$) | 1.06E+08 | 1.60E+08 | 0.711E+08 | 0.948E+08 | 1.52E+08 | 0.924E+08 |
| Error (%) | -20.9 | -19.4 | -46.9 | -29.3 | 13.4 | -31.0 |

(CH: crosshead displacement, Mid: midpoint deflection, TI: Timoshenko beam theory, EB: Euler-Bernoulli beam theory

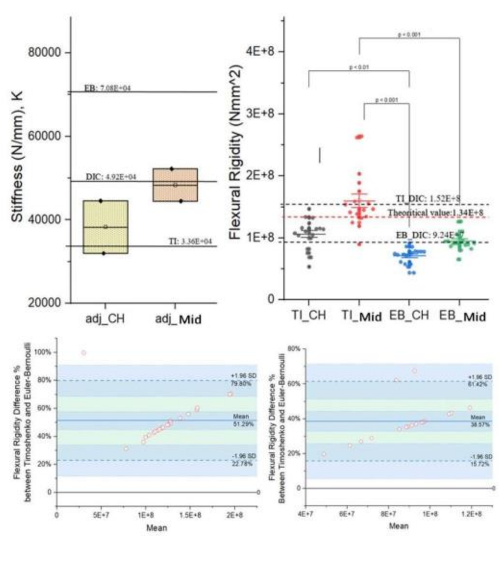


Figure 1. Top Left: Bone surrogate bending stiffness. Top Right: Bone surrogate EI Bottom: Bland-Altman plots show bias (51% and 39%) between TI and EB theories for EI estimation, using crosshead displacement (left) and midpoint deflection (right)
(CH: crosshead displacement, Mid: midpoint deflection, TI: Timoshenko beam theory, EB: Euler-Bernoulli beam theory)