Bader College Undergraduate Student Summer Research Fellowship (USSRF): Project 1 Description

Supervisor: Matthew Haynes

Position: Lecturer of Physical Sciences

Project Title: Generation of exact capillary surface stability curves using elliptic functions

Project Abstract.

In daily life, especially in the UK, we encounter capillary surfaces regularly. These can be raindrops on windows, gas bubbles in our drinks, or in saturated soil. In industry, there are many further uses of capillary surfaces, for example, in inkjet printing or the production of heart stents. The shape of such capillary surfaces is described by the Young—Laplace equation, which relates the curvature of the surface, the surface tension, and the pressure difference across the surface[1]. While this equation has studied for 200 years, only recently have exact solutions been found[2]. These published exact solutions only hold for a small number of physical cases and practical solutions to the Young—Laplace equation are still found using numeric or asymptotic techniques[3].

However, not all solutions to the Young—Laplace equation are physically reasonable, indeed it will produce solutions which, when left in the real world, would destabilize and transition into a different solution of the equation[4]. Determining the stability of the solutions is more challenging than simply finding the solution. However, using a bifurcation technique, it is possible to determine the stability of an entire branch of solutions[5]. Combining modern methods of elliptic integrals to produce exact solutions to the Young—Laplace equation with this bifurcation approach, it will be possible to produce the first exact stability curve[6].

In this project, we will employ elliptic integrals to write exact equations for the shape of a capillary surface. These solutions will then be used to produce an exact stability criteria.

[1] P.S. de Laplace. *Oeuvres comptete de Laplace* 1805
T. Young. An essay on the cohesion of fluids. *Philosophical Transactions of the Royal Society of London,* 1805.

[2] P.G. de Gennes, F Brochard-Wyart, D. Quere. *Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves.* Springer 2004
C. Lv and S. Shi. Wetting states of two-dimensional drops under gravity. *Physical Review* 2018.
H. Cooray, H.E. Huppert, J.A. Neufeld. Maximal liquid bridges between horizontal cylinders. *Proc. R. Soc. A* 2016

[3] P. Concus, R. Finn. The shape of a pendant liquid drop. *Philosophical Transactions of the Royal Society of London* 1979
S. Hartland and R.W. Hartley. *Axisymmetric fluid-liquid interfaces: tables giving the shape of sessile and pendant drops and external menisci, with examples of their use.* Elsevier Science 1976
M. Haynes, S.B.G. O'Brien, E.S. Benilov. Asymptotics of a horizontal liquid bridge. *Physics of fluids* 2016
C. Pozrikidis. Stability of sessile and pendant liquid drops. *Journal of Engineering Mathematics* 2012
M. Pradas, N. Savva, J.B. Benziger, I.G. Kevrekidis, S. Kalliadasis. Dynamics of fattening and thinning 2D sessile droplets. *Langmuir* 2016

[4] R. Finn. *Equilibrium capillary surfaces.* Springer 1986
L.D. Landau, E.M. Lifshitz. *Fluid Mechanics.* Pergamon Press 1987
A.D. Myshkis, V.G. Babskii, N.D. Kopacjevskii, L.A. Slobozhanin, A.D. Tyuptsov, R.S. Wadhwa. Low-gravity fluid mechanics. *Translated from Russian by Wadhwa. Springer-Verlag Berlin* 1987

[5] B.J. Lowry, P.H Steen. Capillary surfaces: stability from families of equilibria with application to the liquid bridge. *Proceedings of the Royal Society of London* 1995
J.H. Maddocks. Stability and folds. *Archive for Rational mechanics and Analysis* 1987

[6] M.Haynes, *Static Capillary Structures and their Stability,* University of Limerick Ph.D. thesis. 2019.

Aims and Outcomes.

On completion of a successful project, I expect the student to have a sufficient understanding of the wider field and a detailed knowledge of the topic. These two aims will culminate in the development of a publication, aimed at the Journal of Fluid Mechanics (JFM). At the end of their onsite period, I expect the student to have completed the fundamental research and have completed an early draft of a summarizing review of current research – ready to be used in an introduction. In the weeks following their onsite period, I expect the student to continue documenting their research culminating in the production a full detailed report. This report will be collated and embellished with my own research to produce a final paper for JFM.

In summary, before arrival I expect the student to:

- Have read and summarized a series of relevant papers

During the student's time onsite I expect the student to:

- Develop exact solutions to the Young—Laplace equation
- Build numerical solutions to the Young—Laplace equation
- Employ bifurcation method to determine the stability of these solutions
- Construct a numerical tool to solve existing stability criteria
- Write an exact stability criteria using elliptic integrals and the bifurcation parameter
- Document their research as they go

After leaving, I expect the student to:

- Collate their research into one document
- Produce a report that contains sufficient detail to be reproducible
- Present their results at the USSRF presentation session

USSRF Learning Outcomes and Skill Development.

During their programme, the student is expected to develop many technical skills. These range from the mathematical (examining a bifurcation parameter in an elliptic function) to the physical (determining the stability threshold of a capillary surface) to the computational (writing basic code which solves the Young—Laplace equation using the shooting method). In addition to these technical skills, the student will also develop various interdisciplinary skills, for example report writing, research summarizing, and importantly time allocation.

The development of these skills will be monitored and assessed through regular meetings. At the start of the project, I would expect these meetings to be almost daily, where in some of the middle weeks and during the writing phase, more autonomy will be expected from the student. I would also expect the length of these meetings to change as the project develops, at the start of the project they might be around 20-30 minutes to give a clear line of enquiry for the student. Then in middle weeks, when the work becomes more technical, these meetings may become longer to allow a greater discussion of the key concepts to ensure that important content is covered rigorously. Finally, in the final weeks of the project we may return to shorter meetings.

Timeline and Milestones.

Prearrival: Collate and summarise relevant literature

Week 1: Produce exact solution for the shape
Week 2: Implement these solutions in MATLAB and examine solution restrictions

Week 3: Develop a numerical solver to verify exact solutions
Week 4: Produce relevant bifurcation diagrams to establish stability criteria
Week 5: Develop a numerical solver to verify stability criteria
Week 6: Use exact solutions to produce exact stability criteria
Week 7: Examine solution space
Week 8: Produce stability diagram

Week 9: Writing – Introduction and Formulation (shape and stability equations, uniqueness parameters)
Week 10 (offsite): Writing -- Exact shape and stability equations
Week 11 (offsite): Writing -- Numerical method and stability curve and conclusions

Students interested in submitting an application for this project should complete the USSRF student application found on the Queen's USSRF website. Applications are due to Traci Allen by March 1st, 2023. See submission instructions posted in the Program Guidelines.